Combinatorial Embeddings and Representations

Thesis

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Combinatorial Embeddings and Representations

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Abstract

Topological embeddings of complete graphs and complete multipartite graphs give rise to combinatorial designs when the faces of the embeddings are triangles. In this case, the blocks of the design correspond to the triangular faces of the embedding. These designs include Steiner, twofold and Mendelsohn triple systems, as well as Latin squares. We look at construction methods, structural properties and other problems concerning these cases.

In addition, we look at graph representations by Steiner triple systems and by combinatorial embeddings. This is closely related to finding independent sets in triple systems. We examine which graphs can be represented in Steiner triple systems and combinatorial embeddings of small orders and give several bounds including a bound on the order of Steiner triple systems that are guaranteed to represent all graphs of a given maximum degree. Finally, we provide an enumeration of graphs of up to six edges representable by Steiner triple systems.

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Lastly but most importantly, I wish to thank my parents who are the reason I've made it this far. Without them, this would not have been possible and so I dedicate this thesis to them.

"As with everything else, so with a mathematical theory:
beauty can be perceived but not explained."
- Arthur Cayley

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CHAPTER 1

Introduction

In this thesis we are concerned with combinatorial embeddings and combinatorial representations. The thesis thus consists of two parts: the five chapters that follow investigate problems concerning embeddings of combinatorial structures while the last three chapters investigate problems concerning graph representations. This chapter serves as an introduction to the concept of each part and to the terminology that will be used throughout the thesis.

The first part of the thesis falls under the area of Topological Graph Theory which is the branch of Graph Theory concerned with surface embeddings of graphs, i.e. graphs that can be drawn on a surface with no edge crossings. It is a well studied area with many theorems and results, see [34]. However, we will focus on triangular embeddings of graphs since this is precisely where the connection between embeddings of graphs and embeddings of combinatorial structures arises. As such, this area of study can be referred to as Topological Design Theory. More details are given later on in this chapter, specifically in Section 1.1.2.

This connection was first observed by Heffter in his paper "Über das Problem der Nachbargebiete" [35] dated November 1890. In this paper Heffter presents a partition of the integers $1, 2, \dots, 12s + 6$, $s \geq 0$ into $4s + 2$ triples so that for each triple $\{a, b, c\}$, $a + b + c \equiv 0 \pmod{12s + 7}$. Then he shows that, under some conditions, these triples can be used to construct a twofold triple system of order $12s+7$ whose blocks are the faces of a triangular embedding of the complete graph K_{12s+7} in an orientable surface. It is still not known if there are infinitely many such values of s but the method is applicable for $s = 0, 1, 2, 4, 5, 11$ and 14, numbers given explicitly in [35]. Another paper of this nature is the one by Emch [18] published in 1929. What makes this paper interesting is that it contains diagrams of the embedding of the twofold triple system of order 6 in the projective plane, the embedding of a pair of Steiner triple systems of order 7 in the torus and the embedding of a pair of Steiner triple systems of order 9 in a pseudosurface formed by a torus. These diagrams are given in the chapters that follow. For more information on Topological Design Theory we refer the reader to a recent survey [22].

In the second part of the thesis we will examine when a graph can be represented by a Steiner triple system or by a topological embedding. Representations of graphs has a long and rich history and we refer the reader to the AMS classification 05C62. Representing graphs by Steiner triple systems relates to finding an independent set in a Steiner triple system. Indeed, representation of arbitrary graphs is a generalization of independent sets. Independent sets have been widely studied in Design Theory, see [8], Chapter 17. In Chapter 7, we will show this relation between representations of graphs and independent sets. In Chapter 8, we provide an enumeration of the number of occurrences of a configuration in a Steiner triple system and in Chapter 9 we extend this idea to representations of graphs by topological embeddings.

1.1 Preliminaries

In this section we provide the terminology that will be used throughout the thesis. Each chapter investigates a different problem and hence additional definitions will be provided where necessary. We will begin with Design Theory terminology and conclude with Topological Graph Theory terminology.

1.1.1 Design Theory

This thesis is mainly concerned with two classes of combinatorial designs, namely, triple systems and Latin squares.

A *triple system* $\mathrm{TS}(n, \lambda)$, is a pair $(V, \mathcal{B})$ where V is a finite set of elements (or points) of cardinality n and $\mathcal{B}$ is a collection of 3-element subsets (the blocks or triples) of V such that every 2-element subset of V occurs in exactly λ blocks of $\mathcal{B}$. A *Steiner triple system of order* n, $\mathrm{STS}(n)$, and a *twofold triple system of order* n, $\mathrm{TTS}(n)$, are triple systems with $\lambda = 1$ and $\lambda = 2$ respectively. An $\mathrm{STS}(n)$ exists if and only if $n \equiv 1$ or $3 \pmod 6$ [37]; such values are called admissible.

Example The unique Steiner triple system of order 7, also known as the Fano plane, consists of the following collection of blocks: $\{0,1,2\}$, $\{0,3,4\}$, $\{0,5,6\}$, $\{1,3,5\}$, $\{1,4,6\}$, $\{2,3,6\}$, $\{2,4,5\}$.

A $\mathrm{TTS}(n)$ can be obtained by combining the block sets of two $\mathrm{STS}(n)$s which have a common point set. Note that two copies of an $\mathrm{STS}(n)$ gives a $\mathrm{TTS}(n)$ with $n(n-1)/6$ repeated blocks. A $\mathrm{TTS}(n)$ with no repeated blocks is said to be *simple.* A simple $\mathrm{TTS}(n)$ exists if and only if $n \equiv 0$ or $1 \pmod 3), n \geq 4$, [12].

Example The unique twofold triple system of order 6 consists of the blocks: $\{0,1,2\}$, $\{0,1,5\}$, $\{0,2,3\}$, $\{0,3,4\}$, $\{0,4,5\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{2,3,5\}$, $\{2,4,5\}$.

A Mendelsohn triple system of order n, $\mathrm{MTS}(n)$, is a triple system defined as above with the only difference that $\mathcal{B}$ is a set of cyclically ordered triples of elements of V which collectively have the property that each ordered pair of elements of V is contained in precisely one triple, i.e. a triple (u, v, w) contains the ordered pairs (u, v), (v, w), (w, u). Such systems exist for $n \equiv 0$ or $1 \pmod 3$, $n \neq 6$ [47].

Example A Mendelsohn triple system of order 7 is given by the blocks: $\{0, 1, 2\}$, $\{0, 2, 1\}$, $\{0, 3, 4\}$, $\{0, 4, 3\}$, $\{0, 5, 6\}$, $\{0, 6, 5\}$, $\{1, 3, 5\}$, $\{1, 6, 3\}$, $\{1, 5, 4\}$, $\{1, 4, 6\}$, $\{2, 5, 3\}$, $\{2, 3, 6\}$, $\{2, 4, 5\}$, $\{3, 6, 4\}$.

An *ℓ-line configuration* in an STS(n) is any collection of ℓ blocks of the Steiner triple system. A configuration is said to be *constant* if every STS(n) contains the same number of copies of the configuration, otherwise it is said to be *variable*. Configurations will be dealt with mostly in Chapter 8. However, we first need to define the well known Pasch configuration which will be used in other chapters as well. The *Pasch configuration* or *quadrilateral* is a 4-line configuration on six distinct points of the form: $\{a, c, d\}$, $\{a, e, f\}$, $\{b, c, e\}$, $\{b, d, f\}$.

Example A Pasch configuration of the Steiner triple system of order 7 in the above example is $\{0, 1, 2\}$, $\{0, 3, 4\}$, $\{1, 3, 5\}$, $\{2, 4, 5\}$.

A subset $S \subseteq V$ in a triple system $T = (V, \mathcal{B})$ is an *independent set* if for all $B \in \mathcal{B}$, $B \nsubseteq S$, i.e. no three points of S occur as a block of $\mathcal{B}$. An independent set S in T is *maximal* if for all $x \in V \setminus S$, $S \cup \{x\}$ is not an independent set in T. On the other hand, it is *maximum* if it has the largest possible cardinality of any independent set in T.

A *transversal design* TD$(3, n)$, of order n and block size 3, is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a $3n$-element set (the points), $\mathcal{G}$ is a partition of V into three parts (the groups) each of cardinality n, and $\mathcal{B}$ is a collection of 3-element subsets (the blocks) of V such that each 2-element subset of V is either contained in exactly one block of $\mathcal{B}$, or in exactly one group of $\mathcal{G}$, but not both. A Latin square of side n determines a TD$(3, n)$ by assigning the row labels, the column labels, and the entries as the three groups of the design. The following example demonstrates this connection between Latin squares and transversal designs.

Example Let $T = \text{TD}(3,3)$ be a transversal design with $V = \mathbb{Z}_9$, $\mathcal{G} = \{0,1,2\}$, $\{3,4,5\}$, $\{6,7,8\}$ and $\mathcal{B} = \{0,3,6\}$, $\{0,4,7\}$, $\{0,5,8\}$, $\{1,3,7\}$, $\{1,4,8\}$, $\{1,5,6\}$, $\{2,3,8\}$, $\{2,4,6\}$, $\{2,5,7\}$. Applying the mappings $0 \to 0_r$, $1 \to 1_r$, $2 \to 2_r$, $3 \to 0_c$, $4 \to 1_c$, $5 \to 2_c$, $6 \to 0_e$, $7 \to 1_e$, $8 \to 2_e$, we obtain the cyclic Latin square of order 3.

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Two triple systems, $(V, \mathcal{B})$ and $(V', \mathcal{B}')$, are said to be *isomorphic* if there exists a bijection $\phi : V \to V'$, such that for each block $B \in \mathcal{B}$, $\phi(B)$ is a block in $\mathcal{B}'$. An isomorphism which maps the system to itself is called an *automorphism*. The set of all automorphisms of a triple system T, with the operation of composition, forms a group called the *full automorphism group* of T and is denoted by $\text{Aut}(T)$. Moreover, a $\text{TS}(n, \lambda)$ is *cyclic* if it has an automorphism of order n. Up to isomorphism, the $\text{STS}(n)$ is unique for $n = 3, 7$ and 9; STS(7) is cyclic. There are two STS(13)s, one of which is cyclic, 80 STS(15)s [9], two of which are cyclic, and there are 11,084,874,829 STS(19)s [36], four of which are cyclic. In terms of the twofold triple systems, the TTS(3) and TTS(6) are unique and are non-simple and simple respectively. There are four TTS(7)s, one of which is simple, and there are 36 TTS(9)s, 13 of which are simple [8]. Finally, there are up to isomorphism, 1, 1, 0, 3, 18, 143 and 4905593 Mendelsohn triple systems of order 3, 4, 6, 7, 9, 10 and 12 respectively [13, 21]. For $n > 12$, no exact value of nonisomorphic $\text{MTS}(n)$s is known.

Similarly, two $\text{TD}(3, n)$s, $(V, \{G_1, G_2, G_3\}, \mathcal{B})$ and $(V', \{G_1', G_2', G_3'\}, \mathcal{B}')$, are said to be *isomorphic* if for some permutation π of $\{1,2,3\}$, there exist bijections $\alpha_i : G_i \to G'_{\pi(i)}$, $i = 1,2,3$, that map blocks of $\mathcal{B}$ to blocks of $\mathcal{B}'$. Two Latin squares are said to be in the same *main class* if the corresponding transversal designs are isomorphic. Up to isomorphism, there is just one main class of each Latin square of order 1, 2 and 3. There are two main classes of Latin squares of order 4 and of order 5. Finally, there are 12, 147, 283657, 19270853541 and 34817397894749939 main classes of Latin squares of order 6, 7, 8, 9 and 10 respectively [46].

1.1.2 Topological Graph Theory

Unless otherwise stated, we will be concerned with closed, connected 2-manifolds (surfaces) with no boundary and with graphs with no loops or multiple edges. There are two types of closed surfaces, orientable and nonorientable. A surface is orientable if the notion of orientation (clockwise or counterclockwise) can be defined consistently on the surface. Any closed orientable surface S_g is topologically equivalent to a sphere with g handles attached to it. For example, the surfaces S_0, S_1 and S_2 are the sphere, the torus and the double torus respectively. Similarly, a surface is nonorientable if there is no way of consistently defining the notion of orientation on the surface and is topologically equivalent to a sphere with γ crosscaps attached to it. It is denoted by N_γ. The surfaces N_1 and N_2 are the projective plane and the Klein bottle respectively.

This thesis will also be concerned with pseudosurfaces. A *pseudosurface* is the topological space which results when finitely many identifications of finitely many points each, are made on a given surface. More precisely, distinct points $\{p_{i,j} : i = 1, 2, \ldots, k,\ j = 0, 1, \ldots, m_i\}$ on a given surface are identified to form points $p_i = \{p_{i,j} : j = 0, 1, \ldots, m_i\},\ i = 1, 2, \ldots, k$ called *singular points* or *pinch points*. The number m_i is the *multiplicity* of the pinch point p_i. It is at these pinch points that a pseudosurface fails to be a 2-manifold.

A surface or a pseudosurface can be illustrated by a polygon, usually a rectangle, where the sides are pairwise identified and each one has a given direction. The surface is obtained by 'gluing' together each pair of identified sides in such a way so that they have the same direction. For example, the torus and the Klein bottle can be illustrated by rectangles as shown in the figure below.

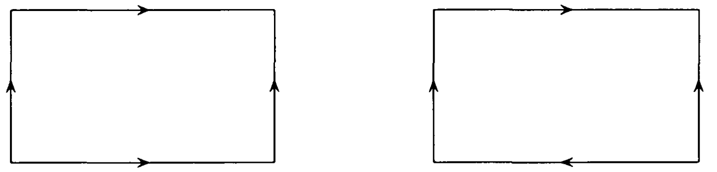


Figure 1.1: The torus and the Klein bottle.

The number of handles g and the number of crosscaps γ is called the *genus* of the orientable and nonorientable surface respectively, $g, \gamma \geq 0$. An *embedding* of a graph G in a surface S_g (or N_γ) is a 'drawing' of G in S_g (or N_γ) such that no pair of edges intersect and where g (or γ) is minimum. A graph embedding divides the surface into a number of connected regions, called *faces*, bounded by edges of the graph. Euler gave a formula relating the number of vertices n, the number of edges e and the number of faces f of a polyhedron: $n - e + f = 2$. This formula was later generalized by Poincaré for any orientable and nonorientable surface: $n - e + f = \chi$ where χ is called the Euler characteristic and is given by $2 - 2g$ if the surface is orientable and $2 - \gamma$ if the surface is nonorientable.

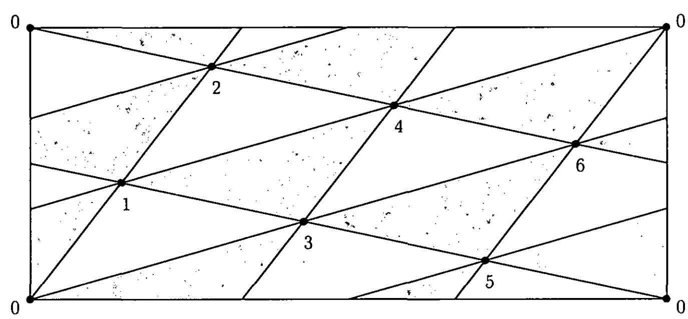
Given a surface embedding of a graph G with vertex set $V(G)$, the *rotation* at a vertex $v \in V(G)$ is the cyclically ordered permutation of vertices adjacent to v, with the ordering determined by the embedding. The set of rotations at all the vertices of G is called the *rotation scheme* for the embedding. In the case of an embedding of G in an orientable surface, the rotation scheme provides a complete description of the embedding. This is not generally the case for a nonorientable surface because the rotation scheme does not enable the faces of the embedding to be unambiguously reconstructed, therefore some additional information is required. However, in the cases we consider this will not be an issue, since extra information which is sufficient to determine the faces will be known.

In [52], Ringel provides a test to determine if a rotation scheme represents a triangular embedding and another one to determine if a triangular embedding is orientable.

Rule Δ: A rotation scheme represents a triangular embedding of a graph G if, for each vertex $a \in V(G)$, whenever the rotation at a contains the sequence $\ldots b\ c \ldots$, then the rotation at b contains either the sequence $\ldots a\ c \ldots$ or the sequence $\ldots c\ a \ldots$.

Rule Δ^*: If the rotations at each vertex can be directed in such a way that for each vertex $a \in V(G)$, whenever the rotation at a contains the sequence $\ldots b\ c \ldots$, then the rotation at b contains the sequence $\ldots c\ a \ldots$, then the embedding is in an orientable surface.

Example The triangular embedding of the complete graph K_7 in the torus is given below together with its rotation scheme. An easy examination shows that the rotation scheme follows both of Ringel's rules.



0:	1	3	2	6	4	5
1:	3	0	5	6	2	4
2:	6	0	3	5	4	1
3:	2	0	1	4	6	5
4:	5	0	6	3	1	2
5:	1	0	4	2	3	6
6:	4	0	2	1	5	3

To see the connection between design theory and graph embeddings, consider the case of an embedding of the complete graph K_n in which all the faces are triangles. In such a triangulation the number of faces around each vertex is $n-1$, and so if $n-1$ is even it may be possible to colour each face using one of two colours, say black or white, so that no two faces of the same colour are adjacent. In this case, we say that the triangulation is *(properly) face two-colourable.* In a face two-colourable triangulation, the set of faces of each colour class form an STS(n). We then say that the two STS(n)s, T_1 and T_2, are *biembedded* in the surface and we denote that biembedding by $T_1 \bowtie T_2$.

A complete graph K_n has a triangulation in an orientable surface if and only if $n \equiv 0, 3, 4$ or $7 \pmod{12}$ and in a nonorientable surface if and only if $n \equiv 0$ or $1 \pmod 3$, $n \neq 3, 4, 7$, [52]. In the orientable case, for $n \equiv 3 \pmod{12}$, the triangulations given by Ringel in [52] using bipartite current graphs, are face two-colourable. For $n \equiv 7 \pmod{12}$, a solution is given by Youngs [58], using what he calls "zigzag diagrams" to again construct bipartite current graphs. The nonorientable case $n \equiv 9 \pmod{12}$ can also be found in [52], and uses another class of bipartite current graphs which Ringel calls "cascades". It is claimed that the method also works for $n \equiv 3 \pmod{12}$, although no details are given. These were later established and presented in Bennett's Ph.D. thesis [3]. A simpler description appears in the survey paper [22]. Somewhat surprisingly, the existence of nonorientable face two-colourable triangulations of K_n for $n \equiv 1 \pmod 6$ was not identified until more recently and was proved by Grannell and Korzhik [32], again using current graphs.

Now consider a face two-colourable triangular embedding of a complete regular tripartite graph $K_{n,n,n}$. In this case, the faces of each colour class can be regarded as the triples of a transversal design TD$(3, n)$, of order n and block size 3, or in other words a Latin square of side n. Similarly as above, we say that the two Latin squares of order n, L_1 and L_2, are biembedded in the surface and we denote the biembedding by $L_1 \bowtie L_2$. It is known that a triangular embedding of a complete regular tripartite graph $K_{n,n,n}$ in a surface is face two-colourable if and only if the surface is orientable [23]. More details regarding biembeddings of Latin squares will be given in Chapter 5.

CHAPTER 2

Biembeddings using the Bose construction

Recall that in a face two-colourable triangulation of K_n, the set of faces of each colour class form a Steiner triple system of order n if and only if $n \equiv 1$ or 3 (mod 6). In this chapter, we seek to identify pairs of STS(n)s, constructed using the well known Bose construction [4], such that the triples of these pairs form a surface when 'glued' together among common edges. This is an alternative approach to using current graphs (see Chapter 1) for constructing face two-colourable triangulations. Indeed, this approach has been successful. In 1978, Ducrocq and Sterboul [17] employed the Bose construction for Steiner triple systems of order $n \equiv 3$ (mod 6) to obtain face two-colourable triangulations of K_n in a nonorientable surface. Later, in 1998, Grannell, Griggs and Širáň [30] also used the Bose construction to do the same in an orientable surface for $n \equiv 3$ (mod 12). Moreover, these face two-colourable triangulations were shown to be isomorphic to those obtained by Ringel using current graphs [52].

The impetus for the work presented here however is a more recent paper by Solov'eva [54]. In this paper, again using the Bose construction, Solov'eva produces nonisomorphic biembeddings of pairs of Steiner triple systems in a nonorientable surface. The purpose of this chapter is threefold. First, by using information about the automorphism group of an STS(n) constructed from the

Bose construction, we determine an exact formula for the number of nonisomorphic nonorientable biembeddings which can be obtained by Solov'eva's method. Secondly, we extend Solov'eva's work to biembeddings of pairs of Steiner triple systems in orientable surfaces. Finally, our approach provides a uniform framework within which both the biembeddings found by Ducrocq and Sterboul [17] and by Grannell, Griggs and Širáň [30] appear.

2.1 Bose construction

In 1939, Bose [4] published a landmark paper on Design Theory in which he presented the following construction for Steiner triple systems of order $n \equiv 3 \pmod 6$. Let G be the cyclic group of order $2t+1$ based on the set $\{0, 1, \ldots, 2t\}$ with addition modulo $2t+1$. Let $X = G \times \{0, 1, 2\}$ and $\mathcal{B}$ be the following collection of triples

$$
\begin{array}{ll}
\text{(A)} & \{(x,0),(x,1),(x,2)\},\ x \in G \\
\text{(B1)} & \{(x,0),(y,0),(z,1)\},\ x,y \in G,\ x \neq y,\ z = (x+y)/2 \\
\text{(B2)} & \{(x,1),(y,1),(z,2)\},\ x,y \in G,\ x \neq y,\ z = (x+y)/2 \\
\text{(B3)} & \{(x,2),(y,2),(z,0)\},\ x,y \in G,\ x \neq y,\ z = (x+y)/2
\end{array}
$$

Then $(X, \mathcal{B})$ is an STS$(6t+3)$. We will denote this system by B.

The Bose construction is capable of numerous generalizations and variations, see for example pages 25 to 27 of [8]. However, the one which is of relevance to this paper is the following which appears in [54] and is ascribed to Levin [39]. With the same base set X as above, let the sets of triples be

$$
\begin{array}{ll}
\text{(A)} & \{(x-\alpha,0),(x,1),(x+\beta,2)\},\ x \in G \\
\text{(B1)} & \{(x,0),(y,0),(z+\alpha,1)\},\ x,y \in G,\ x \neq y,\ z = (x+y)/2 \\
\text{(B2)} & \{(x,1),(y,1),(z+\beta,2)\},\ x,y \in G,\ x \neq y,\ z = (x+y)/2 \\
\text{(B3)} & \{(x,2),(y,2),(z+\gamma,0)\},\ x,y \in G,\ x \neq y,\ z = (x+y)/2
\end{array}
$$

where $\alpha + \beta + \gamma \equiv 0 \pmod{2t+1}$.

Proving that this is a Steiner triple system is straightforward. First observe that the number of blocks is $(2t+1)+3((2t+1)2t/2) = (3t+1)(2t+1)$, precisely the number required. Therefore, it suffices to show that every pair is contained in a block. This is clearly true for all pairs $\{(x,j),(y,j)\}$, $x,y \in G$, $x \neq y$, $j \in \{0,1,2\}$. All pairs $\{(x,0),(z,1)\}$, $x,z \in G$, $z \neq x+\alpha$ are contained in a block of the triples (B1). Similarly, all pairs $\{(x,1),(z,2)\}$ (respectively $\{(x,2),(z,0)\}$), $x,z \in G$, $z \neq x+\beta$ (respectively $z \neq x+\gamma$) are contained in a block of (B2) (respectively (B3)). The remaining pairs $\{(x,0),(x+\alpha,1)\}$, $\{(x,1),(x+\beta,2)\}$, $\{(x,2),(x+\gamma,0)\}$ are contained in a block of the triples (A). We will denote this system by $B_{\alpha,\beta,\gamma}$. Clearly, B is the system $B_{0,0,0}$. Solov'eva then proves the following theorem.

Theorem 2.1.1 (Solov'eva) *The Steiner triple systems B and $B_{\alpha,\beta,\gamma}$ of order $6t+3$ biembed in a nonorientable surface if and only if* $\gcd(\alpha,2t+1) = \gcd(\beta,2t+1) = \gcd(\gamma,2t+1) = 1$.

Proof Consider a point $(x,0)$. In the system B, it occurs in triples with the pair $\{(x,1),(x,2)\}$ and also the following pairs:

$$\{(i,0),((i+x)/2,1)\}, i \neq x, \{(i,2),(-i+2x,2)\}, i \neq x.$$

In the system $B_{\alpha,\beta,\gamma}$, it occurs with the pair $\{(x+\alpha,1),(x-\gamma,2)\}$ and also the pairs:

$$\{(i,0),((i+x)/2+\alpha,1)\}, i \neq x, \{(i,2),(-i+2x-2\gamma,2)\}, i \neq x-\gamma.$$

When the two systems are biembedded, if both α and γ are relatively prime to $2t+1$, the rotation about the point $(x,0)$ is as follows.

$$\begin{aligned}(x,0):\ & \underline{(x,1),(x,2)},\ \underline{(x-2\gamma,2),(x+2\gamma,2)},\ \underline{(x-4\gamma,2),(x+4\gamma,2)},\ldots\\ & \ldots,\ \underline{(x-2t\gamma,2),(x+2t\gamma,2)},\ \underline{(x+\alpha,1),(x+2\alpha,0)},\\ & \underline{(x+2\alpha,1),(x+4\alpha,0)},\ \ldots,\ \underline{(x+2t\alpha,1),(x+4t\alpha,0)}\end{aligned}$$

Pairs which are underlined correspond to triples in the system B. Note that $(x+2t\gamma,2) = (x-\gamma,2)$ and $(x+4t\alpha,0) = (x-2\alpha,0)$. Moreover $\lambda = 2t$ is the least value of λ for which $\lambda\gamma = -\gamma$ and $2\lambda\alpha = -2\alpha$, which guarantees that the rotation is a complete cycle. If either α or γ was not relatively prime to $2t+1$, then this would not be the case and the point $(x, 0)$ would be a pinch point. The proof for the rotation about points $(x, 1)$ and $(x, 2)$ follows similarly.

To prove that the surface is nonorientable assume that the rotation about a point $(x, 0)$ is as above. Then the rotation contains the pairs $(x, 1), (x, 2)$ and $(i - \lambda\alpha, 1), (i, 0)$, $i \neq x$ in that order. Assume that the surface is orientable. Then the rotation about a point $(i, 0)$ contains the pair $(x, 0), (i - \lambda\alpha, 1)$ and therefore also the pair $(i, 2), (i, 1)$ again in that order. Thus, the order of rotation about a point $(i, 0)$, $i \neq x$ is in the opposite direction to that of $(x, 0)$. Now consider another point $(y, 0)$. We have a contradiction and the surface is nonorientable. ■

Extending Solov'eva's work we can also construct a Steiner triple system by reversing the order of the second co-ordinates, i.e. by taking the sets of triples to be

(A) $\{(x - \alpha, 0), (x, 2), (x + \beta, 1)\}$, $x \in G$

(B1) $\{(x, 0), (y, 0), (z + \alpha, 2)\}$, $x, y \in G$, $x \neq y$, $z = (x + y)/2$

(B2) $\{(x, 2), (y, 2), (z + \beta, 1)\}$, $x, y \in G$, $x \neq y$, $z = (x + y)/2$

(B3) $\{(x, 1), (y, 1), (z + \gamma, 0)\}$, $x, y \in G$, $x \neq y$, $z = (x + y)/2$

again where $\alpha + \beta + \gamma \equiv 0 \pmod{2t+1}$. We will denote this system by $B^*_{\alpha,\beta,\gamma}$ and with the additional restriction that the cyclic group G is of order $4t + 1$ prove the following theorem.

Theorem 2.1.2 *If*

(1) $\alpha = 0$ *and* $\gcd(\beta, 4t + 1) = \gcd(\gamma, 4t + 1) = 1$, *or*

(2) $\beta = 0$ *and* $\gcd(\gamma, 4t + 1) = \gcd(\alpha, 4t + 1) = 1$, *or*

(3) $\gamma = 0$ *and* $\gcd(\alpha, 4t + 1) = \gcd(\beta, 4t + 1) = 1$,

then the Steiner triple systems B *and* $B^*_{\alpha,\beta,\gamma}$ *of order* $12t + 3$ *biembed in an orientable surface.*

Proof Consider the first case and assume that $\alpha = 0$ and $\gcd(\beta, 4t+1) = \gcd(\gamma, 4t+1) = 1$. In the system B, a point $(x, 0)$ occurs in triples with the pairs

$$\{(x,1),(x,2)\}, \{(i,0),((i+x)/2,1)\}, i \neq x, \{(i,2),(-i+2x,2)\}, i \neq x,$$

and in the system $B^*_{\alpha,\beta,\gamma}$, it occurs in triples with the pairs

$$\{(x+\beta,1),(x,2)\}, \{(i,0),((i+x)/2,2)\}, i \neq x, \{(i,1),(-i+2x+2\beta,1)\}, i \neq x+\beta.$$

Similarly, a point $(x, 1)$, occurs in the system B in triples with the pairs

$$\{(x,0),(x,2)\}, \{(i,0),(-i+2x,0)\}, i \neq x, \{(i,1),((i+x)/2,2)\}, i \neq x,$$

and in the system $B^*_{\alpha,\beta,\gamma}$, in triples with the pairs

$$\{(x-\beta,0),(x-\beta,2)\}, \{(i,1),((i+x)/2-\beta,0)\}, i \neq x, \{(i,2),(-i+2x-2\beta,2)\},$$
$$i \neq x-\beta.$$

Finally, a point $(x, 2)$, occurs in the system B in triples with the pairs

$$\{(x,0),(x,1)\}, \{(i,1),(-i+2x,1)\}, i \neq x, \{(i,2),((i+x)/2,0)\}, i \neq x,$$

and in the system $B^*_{\alpha,\beta,\gamma}$ in triples with the pairs

$$\{(x,0),(x+\beta,1)\}, \{(i,0),(-i+2x,0)\}, i \neq x, \{(i,2),((i+x)/2+\beta,1)\}, i \neq x.$$

When the two systems B and $B^*_{\alpha,\beta,\gamma}$ are biembedded, the rotation about the points $(x, 0)$, $(x, 1)$, $(x, 2)$ is as follows.

$(x,0)$: $\underline{(x+\beta,1),(x+2\beta,0)}$, $\underline{(x+\beta,2),(x-\beta,2)}$, $\underline{(x-2\beta,0),(x-\beta,1)}$,
$\underline{(x+3\beta,1),(x+6\beta,0)}$, $\underline{(x+3\beta,2),(x-3\beta,2)}$, $\underline{(x-6\beta,0),(x-3\beta,1)}$, ...
..., $\underline{(x+(4t-1)\beta,1),(x+2(4t-1)\beta,0)}$, $\underline{(x+(4t-1)\beta,2),(x-(4t-1)\beta,2)}$,
$\underline{(x-2(4t-1)\beta,0),(x-(4t-1)\beta,1)}$, $\underline{(x,1),(x,2)}$

$(x,1)$: $\underline{(x+2\beta,1),(x+\beta,2)}$, $\underline{(x-3\beta,2),(x-6\beta,1)}$, $\underline{(x-4\beta,0),(x+4\beta,0)}$,
$\underline{(x+10\beta,1),(x+5\beta,2)}$, $\underline{(x-7\beta,2),(x-14\beta,1)}$, $\underline{(x-8\beta,0),(x+8\beta,0)}$, ...
..., $\underline{(x+(4t-7)\beta,1),(x+(4t-3)\beta,2)}$, $\underline{(x-(4t-1)\beta,2),(x-2(4t-1)\beta,1)}$,
$\underline{(x+\beta,0),(x-\beta,0)}$, $\underline{(x-\beta,2),(x-2\beta,1)}$, $\underline{(x-2\beta,0),(x+2\beta,0)}$,
$\underline{(x+6\beta,1),(x+3\beta,2)}$, $\underline{(x-5\beta,2),(x-10\beta,1)}$, $\underline{(x-6\beta,0),(x+6\beta,0)}$,
$\underline{(x+14\beta,1),(x+7\beta,2)}$, ..., $\underline{(x-(4t-3)\beta,2),(x-2(4t-3)\beta,1)}$,

$\underline{(x-(4t-2)\beta,0),(x+(4t-2)\beta,0)},\ \underline{(x+2(4t-1)\beta,1),(x+(4t-1)\beta,2)},$

$\underline{(x,2),(x,0)}$

$(x,2):\underline{(x-2\beta,2),(x-\beta,0)},\ \underline{(x+\beta,0),(x+2\beta,2)},\ \underline{(x+2\beta,1),(x-2\beta,1)},$

$\underline{(x-6\beta,2),(x-3\beta,0)},\ \underline{(x+3\beta,0),(x+6\beta,2)},\ \underline{(x+4\beta,1),(x-4\beta,1)},\ \ldots$

$\ldots,\underline{(x-(4t-3)\beta,2),(x-(4t-1)\beta,0)},\ \underline{(x+(4t-1)\beta,0),(x+2(4t-1)\beta,2)},$

$\underline{(x-\beta,1),(x+\beta,1)},\ \underline{(x,0),(x,1)}.$

As in the proof of Theorem 2.1.1, pairs which are underlined correspond to triples in the system B. Note that $\beta \equiv -\gamma \pmod{4t+1}$ since $\alpha = 0$. Moreover, $\lambda = 4t$ is the least value of λ for which $\lambda\beta = -\beta$, which guarantees that the rotation is a complete cycle. If either β or γ was not relatively prime to $4t+1$, then this would not be the case and the points $(x,0)$, $(x,1)$ and $(x,2)$ would be pinch points. To prove that the surface is orientable it suffices to show that Ringel's Rule Δ^* holds. An easy but tedious examination shows that the above rotations form an orientable triangular biembedding of the systems B and $B^*_{\alpha,\beta,\gamma}$. ■

2.2 Automorphisms

2.2.1 Nonorientable biembeddings

In this section, we determine the automorphism group of the nonorientable biembeddings obtained using the Bose construction and prove a formula for the number of such biembeddings. Recall that we use the notation $B \bowtie B_{\alpha,\beta,\gamma}$ to denote the biembedding, in this case nonorientable, of the system B with the system $B_{\alpha,\beta,\gamma}$ with the triples of B coloured black and the triples of $B_{\alpha,\beta,\gamma}$ coloured white. In order to obtain our results we will need to know the automorphism group of B. This was determined by Lovegrove [40]. As is well known, in the basic Bose construction the group G need not be cyclic but can be any Abelian group of odd order. Lovegrove divides the automorphisms into two types, *standard* and *non-standard* though the distinction between the two need not concern us here. In this context, he proves Theorem 2.2.1.

Before we state the theorem we need the following definitions. Given two groups G and H and a group homomorphism $\theta : H \to Aut(G)$, the group $G \rtimes_\theta H$ is called the *semidirect product* of G by H with underlying set the cartesian product $G \times H$ and group operator $*$: $(g_1, h_1) * (g_2, h_2) = (g_1\theta(h_1)(g_2), h_1h_2)$, where g_1, $g_2 \in G$, h_1, $h_2 \in H$. If $H \cong Aut(G)$ and θ is the identity, then the semidirect product is called the *holomorph* of G denoted by $\mathrm{Hol}(G)$.

Theorem 2.2.1 (Lovegrove) *The group of standard automorphisms of the Steiner triple system constructed from an odd order Abelian group G is isomorphic to $Hol(G) \times \mathbb{C}_3$ and so is of order $3|G||Aut(G)|$.*

With regard to nonstandard automorphisms, Lovegrove shows that these occur only if the group G is of the form $\mathbb{C}_3^n \times \mathbb{C}_9^m$, $n+m \neq 0$. Therefore, the only Steiner triple systems obtained from the Bose construction using a cyclic group which have nonstandard automorphisms are the STS(9) obtained from $\mathbb{C}_3$ and the STS(27) from $\mathbb{C}_9$. These two exceptions will cause us no problems and we will deal with them later. Hence, for all other systems obtained from cyclic groups the group given in the above theorem is the full automorphism group of the system B. It further follows that $\mathrm{Aut}(B)$ is generated by the three following mappings:

1. $(i, j) \mapsto (i + 1, j)$,
2. $(i, j) \mapsto (\lambda i, j)$ where $\gcd(\lambda, 2t + 1) = 1$,
3. $(i, j) \mapsto (i, j + 1)$.

Automorphisms of the biembedding $B \bowtie B_{\alpha,\beta,\gamma}$ will be of two types, those that preserve the colour classes and those which reverse them. We first consider the colour preserving automorphisms. Any such automorphism will belong to $\mathrm{Aut}(B)$ and therefore we consider the action of the three generators.

1. The mapping $(i, j) \mapsto (i + 1, j)$ stabilizes the biembedding $B \bowtie B_{\alpha,\beta,\gamma}$ and is therefore a colour preserving automorphism. Denote this automorphism by τ.

2. The mapping $(i, j) \mapsto (\lambda i, j)$ maps the biembedding $B \bowtie B_{\alpha,\beta,\gamma}$ to $B \bowtie B_{\lambda\alpha,\lambda\beta,\lambda\gamma}$.

3. The mapping $(i, j) \mapsto (i, j + 1)$ maps the biembedding $B \bowtie B_{\alpha,\beta,\gamma}$ to $B \bowtie B_{\gamma,\alpha,\beta}$.

Turning now to colour reversing automorphisms, let σ be the mapping defined by (a) $(i, 0) \mapsto (-i, 0)$, (b) $(i, 1) \mapsto (-i + \alpha, 1)$ and (c) $(i, 2) \mapsto (-i - \gamma, 2)$. Then, as is easily verified, σ maps the biembedding $B \bowtie B_{\alpha,\beta,\gamma}$ to $B_{\alpha,\beta,\gamma} \bowtie B$, i.e. it reverses the colours of the two systems.

In any biembedding, either all automorphisms are colour preserving or there are equal numbers which are colour preserving and colour reversing. The mapping τ has order $2t + 1$ and σ has order 2. Moreover, $\sigma\tau = \tau^{2t}\sigma$, i.e. τ and σ generate the dihedral group $\mathbb{D}_{2t+1}$ of order $4t + 2$ which is the full automorphism group of any biembedding $B \bowtie B_{\alpha,\beta,\gamma}$.

Next we investigate the number of nonisomorphic nonorientable biembeddings which are obtained from the Bose construction. From the above, we have already established that the biembedding $B \bowtie B_{\alpha,\beta,\gamma}$ is isomorphic to $B \bowtie B_{\lambda\alpha,\lambda\beta,\lambda\gamma}$ where $\gcd(\lambda, 2t+1) = 1$. Therefore, $B \bowtie B_{\alpha,\beta,\gamma}$ is isomorphic to $B \bowtie B_{1,q,r}$ where $q = \beta\alpha^{-1}$ and $r = \gamma\alpha^{-1}$. As a first step we count the number of biembeddings $B \bowtie B_{1,q,r}$ where $1+q+r \equiv 0 \pmod{2t+1}$ and $\gcd(q, 2t+1) = \gcd(r, 2t+1) = 1$. Equivalently we require the cardinality of the set

$$\{q : \gcd(q, 2t + 1) = \gcd(q + 1, 2t + 1) = 1, 1 \le q \le 2t - 1\}.$$

This is a generalization of the well-known Euler's ϕ-function. Let $N \ge 2$ be an integer. Then for any integer k such that $1 \le k \le N - 1$ define $\phi_k(N)$ to be the number of consecutive sequences of k integers $q, q + 1, \ldots, q + k - 1$, $1 \le q \le N-k$, all of which are coprime to N. Then ϕ_k is a multiplicative function, i.e. if $\gcd(N, M) = 1$, then $\phi_k(NM) = \phi_k(N)\phi_k(M)$. Let $N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ be the prime factorization of N. Then to compute the value of $\phi_k(N)$, it suffices to know the value of $\phi_k(p_i^{\alpha_i})$ for each i. This is straightforward: if p is prime then

$\phi_k(p) = p - k$, and if $\alpha \geq 2$ then $\phi_k(p^\alpha) = (p-k)p^{\alpha-1}$ if $k \leq p$ and 0 otherwise. For our purposes, the number of biembeddings $B \bowtie B_{1,q,r}$ where $1 + q + r \equiv 0 \pmod{2t+1}$ is given by $\phi_2(2t+1)$.

It remains to consider the action of the mapping $(i,j) \mapsto (i, j+1)$ on this collection of biembeddings. Applying the mapping to the biembedding $B \bowtie B_{1,q,r}$ and multiplying the subscripts by q^{-1} in order to restore the first subscript to unity, gives the biembedding $B \bowtie B_{1,rq^{-1},q^{-1}}$ and applying the mapping again and rescaling gives $B \bowtie B_{1,r^{-1},qr^{-1}}$. If $r = q^2$ so that $1 + q + q^2 \equiv 0 \pmod{2t+1}$, the three biembeddings are identical, but otherwise they are not. Define $\psi(2t+1)$ to be the cardinality of the set

$$\{q : 1 + q + q^2 \equiv 0 \pmod{2t+1}, \gcd(q, 2t+1) = 1, 1 \leq q \leq 2t-1\}.$$

Then the number of nonisomorphic biembeddings $B \bowtie B_{\alpha,\beta,\gamma}$ is

$$(\phi_2(2t+1) - \psi(2t+1))/3 + \psi(2t+1) = (\phi_2(2t+1) + 2\psi(2t+1))/3.$$

We now show how to calculate the value of the function ψ.

Let $N \geq 2$ be an integer and $N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ be its prime factorization. Then by the Chinese Remainder Theorem, the number of solutions of the congruence $1 + x + x^2 \equiv 0 \pmod N$ is the product of the number of solutions to the same congruence modulo $p_i^{\alpha_i}$ for each $p_i^{\alpha_i}$ dividing N. There are no solutions for $p = 2$ and $p^2 = 9$ but one solution for $p = 3$. If $p \geq 5$ is prime then there are no solutions for $p \equiv 5 \pmod 6$ and two solutions for $p \equiv 1 \pmod 6$. Hence there are no solutions to $1 + x + x^2 \equiv 0 \pmod N$ for $N = 2^\alpha$, $\alpha \geq 1$; or $N = 3^\alpha$, $\alpha \geq 2$; or $N = p^\alpha$, $p \equiv 5 \pmod 6$ and prime, $\alpha \geq 1$.

It remains to consider the case $N = p^\alpha$, $p \equiv 1 \pmod 6$ and prime, $\alpha \geq 1$. Now by Theorem 68, page 115 of [49], there are three solutions to the congruence $x^3 \equiv 1 \pmod{p^\alpha}$ for $p \equiv 1 \pmod 6$, $\alpha \geq 1$, one of which is $x = 1$ and is not a solution to $1 + x + x^2 \equiv 0 \pmod{p^\alpha}$. As a result, there are two solutions to the congruence $1 + x + x^2 \equiv 0 \pmod{p^\alpha}$ in this case. Note that all solutions to $1 + x + x^2 \equiv 0 \pmod N$ necessarily have $\gcd(x, \gamma) = 1$, as any common factor of x and γ would divide $x + x^2$ and hence also divide 1. Thus, we now have the following theorem.

Theorem 2.2.2 *Let* $n = 6t+3$. *Then the number of nonisomorphic nonorientable biembeddings* $B \bowtie B_{\alpha,\beta,\gamma}$ *of a pair of Steiner triple systems constructed using the Bose construction from the cyclic group* $\mathbb{C}_{2t+1}$ *is*

- $1,\ $ *if* $n = 9$;
- $n/27,\ $ *if* $n = 3^\alpha,\ \alpha \geq 3$;
- $n\Pi_{i=1}^r(1 - 2/p_i)/27 + \beta,\ $ *if* $n = 3^\alpha\Pi_{i=1}^r p_i^{\alpha_i},\ p_i > 3$ *is prime,*

  where $\beta = 0$ *if* $\alpha \geq 3$ *or any* $p_i \equiv 5 \pmod 6$

  and $\beta = 2^{r+1}/3$ *if* $\alpha = 1$ *or* 2, *and all* $p_i \equiv 1 \pmod 6$.

Proof First, we deal with the two exceptional cases $n = 9$ and $n = 27$ for which the STS(n) has nonstandard automorphisms.

If $n = 9$, there is a unique biembedding of a pair of STS(9)s, see page 139 of [22]. It is the biembedding $B \bowtie B_{1,1,1}$.

If $n = 27$, there are precisely three biembeddings $B \bowtie B_{1,q,r}$, namely $(q,r) = (1,7),\ (4,4),\ (7,1)$ which are isomorphic by standard automorphisms.

If $n = 6t + 3 = 3^\alpha,\ \alpha \geq 4$. Then $\phi_2(2t + 1) = 3^{\alpha-2}$ and $\psi(2t + 1) = 0$. Therefore the number of nonisomorphic biembeddings $B \bowtie B_{\alpha,\beta,\gamma}$ is $\phi_2(2t + 1)/3 = n/27$.

Let $n = 6t+3 = 3^\alpha\Pi_{i=1}^r p_i^{\alpha_i}$, $p_i > 3$ is prime. Then $\phi_2(2t+1) = 3^{\alpha-2}\Pi_{i=1}^r p_i^{\alpha_i-1}(p_i - 2) = n\Pi_{i=1}^r(1 - 2/p_i)/9$. In addition to the discussion above $\psi(2t + 1) = 0$ if $\alpha - 1 \geq 2$ or any $p_i \equiv 5 \pmod 6$ and 2^r otherwise. Put $\beta = 2\psi(2t + 1)/3$, then the number of nonisomorphic biembeddings is $(\phi_2(2t + 1) + 2\psi(2t + 1))/3 = n\Pi_{i=1}^r(1 - 2/p_i)/27 + \beta$. ■

Finally in this section we identify the particular biembedding of Ducrocq and Sterboul [17]. The two Steiner triple systems which they give, using the cyclic group G of order $2t+1$ based on the set $\{0, 1, \ldots, 2t\}$ with addition modulo $2t+1$, and with base set $X = G \times \{0, 1, 2\}$ consist of the following collections of triples $\mathcal{B}_0$,

(A) $\{(x,0),(x,1),(x+2t,2)\},\ x \in G$

(B1) $\{(x,0),(y,0),(z,1)\},\ x,y \in G,\ x \neq y,\ z=(x+y)/2$

(B2) $\{(x,1),(y,1),(z+2t,2)\},\ x,y \in G,\ x \neq y,\ z=(x+y)/2$

(B3) $\{(x+2t,2),(y+2t,2),(z,0)\},\ x,y \in G,\ x \neq y,\ z=(x+y)/2$

and of the following collections of triples $\mathcal{B}_1$,

(A) $\{(x,0),(x+1,1),(x+2t-1,2)\},\ x \in G$

(B1) $\{(x,0),(y,0),(z+1,1)\},\ x,y \in G,\ x \neq y,\ z=(x+y)/2$

(B2) $\{(x+1,1),(y+1,1),(z+2t-1,2)\},\ x,y \in G,\ x \neq y,\ z=(x+y)/2$

(B3) $\{(x+2t-1,2),(y+2t-1,2),(z,0)\},\ x,y \in G,\ x \neq y,\ z=(x+y)/2$

Applying the mapping (a) $(i,0) \mapsto (i,0)$, (b) $(i,1) \mapsto (i,1)$, and (c) $(i,2) \mapsto (i+1,2)$ to the set X, the Steiner triple system $(X,\mathcal{B}_0)$ becomes the system B and $(X,\mathcal{B}_1)$ becomes the system $B_{1,2t-1,1}$. Hence the biembedding is $B \bowtie B_{1,1,2t-1}$.

2.2.2 Orientable biembeddings

In this section, we apply the same process as in the previous section to determine the automorphism group of the orientable biembeddings obtained using the Bose construction. We also show that in this case all such biembeddings are in fact isomorphic. Thus, the biembedding constructed by Grannell, Griggs and Širáň [30] is the unique biembedding of its type. Again we will use the notation $B \bowtie B^*_{\alpha,\beta,\gamma}$ to denote that the system B biembeds, in this case orientably, with the system $B^*_{\alpha,\beta,\gamma}$, with the triples of B coloured black and the triples of $B^*_{\alpha,\beta,\gamma}$ coloured white.

As before, for colour preserving automorphisms, we consider the action of the three mappings which generate $\mathrm{Aut}(B)$, i.e. (1) $(i,j) \mapsto (i+1,j)$, (2) $(i,j) \mapsto (\lambda i,j)$ where $\gcd(\lambda, 4t+1)=1$, and (3) $(i,j) \mapsto (i,j+1)$.

1. The mapping $(i,j) \mapsto (i+1,j)$ stabilizes the biembedding $B \bowtie B^*_{\alpha,\beta,\gamma}$ and is therefore a colour preserving automorphism. It is also orientation-preserving. Again we denote this automorphism by τ.

2. The mapping $(i,j) \mapsto (\lambda i, j)$ maps the biembedding $B \bowtie B^*_{\alpha,\beta,\gamma}$ to $B \bowtie B^*_{\lambda\alpha,\lambda\beta,\lambda\gamma}$.

3. The mapping $(i,j) \mapsto (i, j+1)$ maps the biembedding $B \bowtie B^*_{\alpha,\beta,\gamma}$ to $B \bowtie B^*_{\beta,\gamma,\alpha}$.

It now follows immediately that all the orientable biembeddings of Theorem 2.1.2, i.e. $B \bowtie B^*_{0,\beta,-\beta}$, $B \bowtie B^*_{-\gamma,0,\gamma}$, $B \bowtie B^*_{\alpha,-\alpha,0}$ where $\gcd(\alpha, 4t+1) = \gcd(\beta, 4t+1) = \gcd(\gamma, 4t+1)$ are isomorphic. Hence, the biembedding of Grannell, Griggs and Širáň in [30] is the unique biembedding of this type and we will represent it in the standard from $B \bowtie B^*_{1,0,-1}$.

Turning to colour reversing automorphisms, let ρ be the mapping defined by (a) $(i,0) \mapsto (-i,0)$, (b) $(i,1) \mapsto (-i+1,2)$ and (c) $(i,2) \mapsto (-i+1,1)$. Then, as is easily verified, ρ maps the biembedding $B \bowtie B^*_{1,0,-1}$ to $B^*_{1,0,-1} \bowtie B$, i.e. it reverses the colours of the two systems. It also reverses the orientation. Similarly to the nonorientable case, ρ has order 2 and τ and ρ generate the dihedral group $\mathbb{D}_{4t+1}$ of order $8t+2$ which is the full automorphism group of the biembedding. The subgroups of all colour preserving and of all orientation-preserving automorphisms are both the cyclic group $\mathbb{C}_{4t+1}$.

CHAPTER 3

Recursive constructions of triangulations with one pinch point

This chapter deals with recursive constructions of face two-colourable triangulations with one pinch point of the complete graph K_n in an orientable surface. As already mentioned in the introduction, triangulations of the complete graph K_n in an orientable surface exist when $n \equiv 0, 3, 4$ or $7 \pmod{12}$ [52] and they may be face two-colourable only when $n \equiv 3$ or $7 \pmod{12}$ [52, 58]. Then, each set of faces of each colour in a face two-colourable triangulation of K_n forms an STS(n). However, Steiner triple systems of order n exist for all $n \equiv 1$ or $3 \pmod{6}$ [37], i.e. $n \equiv 1, 3, 7$ or $9 \pmod{12}$. Since face two-colourable orientable biembeddings of pairs of STS(n)s exist when $n \equiv 3$ or $7 \pmod{12}$ we focus on the case $n \equiv 1 \pmod{12}$ and consider the question of how close it is possible to biembed a pair of STS(n)s of these orders in an orientable surface.

From Euler's formula, we know that biembeddings of pairs of STS(n)s, where $n \equiv 1 \pmod{12}$, are nonorientable, i.e. the Euler characteristic is odd. Consider the situation where one of the points in such a biembedding is a regular pinch point of multiplicity 2, i.e. the rotation about it will consist of two cycles of equal length $(n-1)/2$. If such a biembedding exists then it may be orientable since the addition of an extra point changes the Euler characteristic to even. In a sense these pseudosurfaces are as close to being orientable surfaces as is possible. We prove the existence of these embeddings for all $n \equiv 13$ or $37 \pmod{72}$. However, in order to prove our general result we first need a biembedding of the smallest value $n = 13$ which we present in the next section together with some computational results.

3.1 Biembeddings of STS(13)s

Up to isomorphism, there exist precisely two STS(13)s, [14]. One of these has a cyclic automorphism and can be constructed on the base set $\mathbb{Z}_{13}$ by the action of the group generated from the mapping $i \mapsto i+1 \pmod{13}$ on the two starter blocks $\{0,1,4\}$ and $\{0,2,7\}$. Denote this system by S. The system also has a further automorphism $\tau : i \mapsto 3i \pmod{13}$ of order 3 giving the full automorphism group of order 39. Without loss of generality we can assume that the point 0 is the pinch point. We applied all $12!/2^6 6! = 10395$ involutions with no fixed points on the set $\mathbb{Z}_{13} \setminus \{0\}$ in turn to S. Each involution produced a Steiner triple system S' isomorphic to S. We then tested whether S biembedded with S' in an orientable pseudosurface with 0 as a regular pinch point of multiplicity 2.

There are only two involutions which produce such biembeddings. These are $\sigma = (1\ 10)(2\ 8)(3\ 4)(5\ 7)(6\ 11)(9\ 12)$ and $\sigma' = (1\ 12)(2\ 8)(3\ 10)(4\ 9)(5\ 7)(6\ 11)$. The full automorphism group of both of the biembeddings is the cyclic group $\mathbb{C}_6$, generated by the permutations $g = \sigma\tau = \tau\sigma = (1\ 4\ 9\ 10\ 3\ 12)(2\ 11\ 5\ 8\ 6\ 7)$ and $g' = \sigma'\tau = \tau\sigma' = (1\ 10\ 9\ 12\ 3\ 4)(2\ 11\ 5\ 8\ 6\ 7)$ respectively. Automorphisms of even order are colour reversing and those of odd order are colour preserving. All automorphisms are orientation-preserving. However, the two biembeddings are nonisomorphic. This can be proved by counting the Pasch configurations in each TTS(13); there are 112 and 82 Pasch configurations respectively.

Below we give the rotation schemes of these biembeddings together with the voltage graphs from which the biembeddings can be derived from. Voltage graphs are used in Topological Graph Theory to represent embeddings of large graphs.

A *voltage graph* G is a directed graph whose edges are labelled by elements of a finite group H; the labels are called voltages and H the voltage group. Note that edges with no label or direction have zero voltage. Then the derived graph has vertex set $V(G) \times H$ and edge set $E(G) \times H$.

0:	1	4	9	10	3	12 |	2	7	6	8	5	11
1:	4	0	12	6	10	11	5	2	8	3	9	7
2:	7	0	11	12	9	4	10	8	1	5	6	3
3:	12	0	10	5	4	7	2	6	11	9	1	8
4:	0	1	7	3	5	8	11	6	12	10	2	9
5:	11	0	8	4	3	10	12	7	9	6	2	1
6:	8	0	7	10	1	12	4	11	3	2	5	9
7:	0	2	3	4	1	9	5	12	11	8	10	6
8:	0	6	9	12	3	1	2	10	7	11	4	5
9:	10	0	4	2	12	8	6	5	7	1	3	11
10:	0	9	11	1	6	7	8	2	4	12	5	3
11:	0	5	1	10	9	3	6	4	8	7	12	2
12:	0	3	8	9	2	11	7	5	10	4	6	1

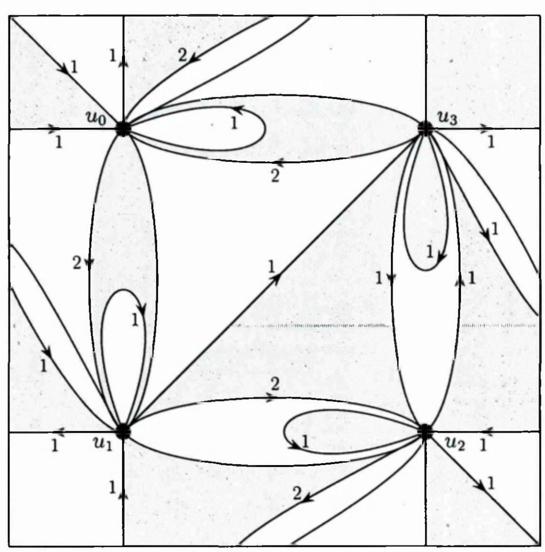


Figure 3.1: Toroidal embedding of voltage graph of biembedding #1.

The voltages in the above voltage graph are taken in the group $\mathbb{Z}_3 = \{0, 1, 2\}$. Therefore, the embedding derived from the above voltage graph has vertex set $\{u_0^0, u_0^1, u_0^2, u_1^0, u_1^1, u_1^2, u_2^0, u_2^1, u_2^2, u_3^0, u_3^1, u_3^2\}$. Additionally, the embedding has 20 black triangular faces, 20 white triangular faces and two open faces bounded by two disjoint 6-cycles. The black triangular faces are $(u_2^0 u_2^1 u_2^2)$, $(u_3^0 u_3^1 u_3^2)$, $(u_0^i u_1^i u_1^{i+2})$,

$(u_0^i u_0^{i+1} u_3^{i+1})$, $(u_0^{i+1} u_2^i u_3^i)$, $(u_1^i u_2^{i+1} u_3^{i+2})$, $(u_0^i u_1^{i+1} u_2^{i+1})$, $(u_1^i u_2^{i+2} u_3^{i+1})$, the white triangular faces are $(u_0^0 u_0^1 u_0^2)$, $(u_1^0 u_1^1 u_1^2)$, $(u_0^i u_2^i u_3^i)$, $(u_0^i u_1^i u_3^{i+1})$, $(u_0^i u_1^{i+1} u_2^{i+2})$, $(u_0^i u_1^{i+2} u_2^{i+2})$, $(u_1^i u_2^i u_2^{i+2})$, $(u_2^i u_3^{i+1} u_3^{i+2})$, and the 6-cycles are $(u_0^0 u_2^0 u_0^2 u_2^2 u_0^1 u_2^1)$, $(u_1^0 u_3^0 u_1^1 u_3^1 u_1^2 u_3^2)$. Let $u_0^0 = 1$, $u_0^1 = 3$, $u_0^2 = 9$, $u_1^0 = 2$, $u_1^1 = 6$, $u_1^2 = 5$, $u_2^0 = 4$, $u_2^1 = 12$, $u_2^2 = 10$, $u_3^0 = 7$, $u_3^1 = 8$, $u_3^2 = 11$. Finally, by adding the point 0 to the embedding and connecting it to the 12 points gives the biembedding of a pair of STS(13)s in an orientable surface with one pinch point with the rotation scheme given above. A similar approach gives the second biembedding from the voltage graph below.

0:	1	4	3	12	9	10 | 7	2	11	5	8	6	
1:	4	0	10	11	12	6	8	3	9	7	5	2
2:	0	7	9	4	1	5	6	3	8	10	12	11
3:	12	0	4	7	10	5	11	9	1	8	2	6
4:	0	1	2	9	8	5	12	10	6	11	7	3
5:	0	11	3	10	9	6	2	1	7	12	4	8
6:	0	8	1	12	3	2	5	9	11	4	10	7
7:	2	0	6	10	3	4	11	8	12	5	1	9
8:	6	0	5	4	9	12	7	11	10	2	3	1
9:	10	0	12	8	4	2	7	1	3	11	6	5
10:	0	9	5	3	7	6	4	12	2	8	11	1
11:	5	0	2	12	1	10	8	7	4	6	9	3
12:	0	3	6	1	11	2	10	4	5	7	8	9

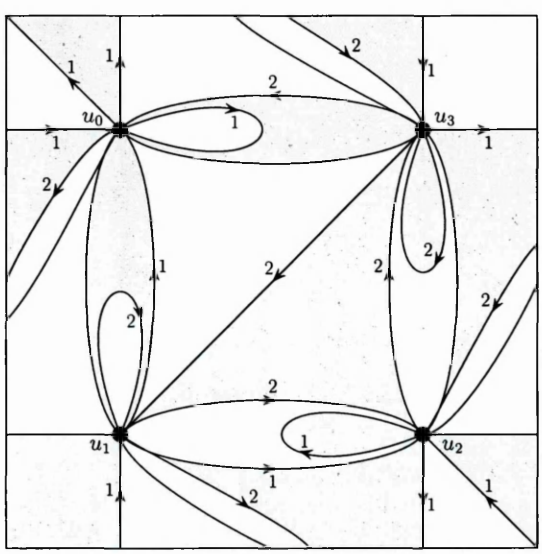


Figure 3.2: Toroidal embedding of voltage graph of biembedding #2.

For the other STS(13) we take the representation given in [44] and for the sake of completeness list the triples, omitting the set brackets for clarity.

$$123, 145, 167, 189, 1ab, 1cd, 246, 257, 28a, 29c, 2bd, 348, 35c$$
$$36d, 37b, 39a, 479, 4ad, 4bc, 56a, 58b, 59d, 68c, 69b, 78d, 7ac$$

The full automorphism group is the dihedral group $\mathbb{D}_3$ of order 6 generated by the permutations $(1\ 2\ 8)(3\ a\ 9)(4\ b\ c)(5\ d\ 6)$ and $(1\ 5)(2\ 6)(3\ a)(8\ d)(b\ c)$. The automorphism partitioning is $\{1, 2, 5, 6, 8, d\}, \{3, 9, a\}, \{4, b, c\}, \{7\}$. Without loss of generality, there are therefore four possibilities for the pinch point, namely, 1, 3, 4, and 7. We considered each of these in turn applying the 10395 involutions without fixed points as in the case of the cyclic STS(13) above. The results are summarized below.

Pinch point 1.

There are three permutations which give biembeddings.

These are $(2\ 5)(3\ d)(4\ c)(6\ b)(7\ 8)(9\ a)$, $(2\ 6)(3\ c)(4\ b)(5\ 9)(7\ d)(8\ a)$, and $(2\ c)(3\ 9)(4\ a)(5\ 6)(7\ b)(8\ d)$. Each TTS(13), obtained by the biembedding, contains 78, 82 and 98 Pasch configurations respectively and therefore the biembeddings are nonisomorphic.

Pinch point 3.

There are two permutations which give biembeddings.

These are $(1\ d)(2\ 8)(4\ 6)(5\ 9)(7\ c)(a\ b)$ and $(1\ c)(2\ 6)(4\ 7)(5\ d)(8\ a)(9\ b)$ but the two biembeddings are isomorphic under the permutation $(1\ 6)(2\ d)(4\ c)(5\ 8)(9\ a)$. Both contain 106 Pasch configurations.

Pinch point 4.

There are four permutations which give biembeddings.

These are $(1\ b)(2\ c)(3\ 7)(5\ 6)(8\ d)(9\ a)$, $(1\ b)(2\ 7)(3\ c)(5\ 8)(6\ a)(9\ d)$, $(1\ d)(2\ 3)(5\ c)(6\ 7)(8\ 9)(a\ b)$, and $(1\ 2)(3\ 9)(5\ c)(6\ b)(7\ a)(8\ d)$. They contain 116, 80, 80 and 116 Pasch configurations respectively. The biembeddings given by permutations #1 and #4, and those by permutations #2 and #3 are isomorphic under the permutation $(1\ 5)(2\ 6)(3\ a)(8\ d)(b\ c)$.

Pinch point 7.

There are no biembeddings.

These six biembeddings have only the identity and the involution given as automorphisms.

Next we give a recursive construction $n \to 3n-2$ for these embeddings. This construction is a modification of the one given in [27]. Note that we will refer to face two-colourable orientable triangular embeddings of the complete graph K_n instead of biembedded pairs of Steiner triple systems of order n; the two of course are equivalent.

Theorem 3.1.1 *Let* $n \equiv 1 \pmod{12}$. *If the complete graph* K_n *has a face two-colourable orientable triangular embedding in a pseudosurface with one regular pinch point of multiplicity 2, then the complete graph* K_{3n-2} *also has a face two-colourable orientable triangular embedding in a pseudosurface with one regular pinch point of multiplicity 2.*

Proof Let η be a face two-colourable triangular embedding of K_n in an orientable pseudosurface S with one regular pinch point of multiplicity 2 which we will denote by ∞. Let the triangular faces of the embedding be properly coloured black and white and let a fixed orientation of the surface be chosen. Consider the restricted embedding of the graph $G = K_n - \infty \simeq K_{n-1}$ which will be obtained in the following way. Firstly, remove from the original embedding η of K_n the pinch point ∞, all open arcs that correspond on the surface to edges incident with ∞ and all open triangular faces of the embedding that correspond to triangles originally incident with the point ∞. By doing that, we create two *holes* in our surface. The resulting *bordered surface* has a face two-colourable triangular embedding $\phi : G \to S$. The boundaries of the two holes in S have the form $\phi(D_k)$, $k = 0, 1$, where D_k are disjoint cycles in G each of length $(n-1)/2$.

Now take three disjoint copies of the embedding ϕ including the proper two-colouring of triangular faces inherited from η. More precisely, for each $i \in \mathbb{Z}_3$ we take a complete graph G^i of order $n-1$ and a face two-colourable triangular embedding $\phi^i : G^i \to S^i$ of G^i in a bordered clockwise oriented surface S^i such that the natural mapping $f^i : G \to G^i$ which assigns the superscript i to each vertex of G is a colour-preserving and orientation-preserving isomorphism of the triangular embeddings ϕ and ϕ^i. We assume that the surfaces S^i are mutually disjoint. The embedding ϕ has $t = n(n-1)/6 - (n-1)/2 = (n-1)(n-3)/6$ white triangular faces. Let $\mathcal{T}$ be the set of these faces and let $\mathcal{T}^i = f^i(\mathcal{T})$ be the corresponding set of all white triangular faces in the embedding ϕ^i for $i \in \mathbb{Z}_3$. In what follows we describe a procedure which, when carried out successively for each $T \in \mathcal{T}$, will merge the bordered surfaces S^i in a way suitable for our purposes.

Let u, v, w be vertices of G such that u, v, w are corners of a fixed white triangular face T of ϕ; we may without loss of generality assume that the chosen clockwise orientation of S induces the cyclic permutation (uvw) of vertices of the face T. For this particular T, consider the auxiliary face-two-coloured embedding ψ_T of the complete tripartite graph $K_{3,3,3}$ in a torus with three holes cut in its surface, as depicted in Figure 3.3 where the holes are depicted as the diagonally hatched regions. The three vertex-parts of our $K_{3,3,3}$ in Figure 3.3 consist of vertices u_T^i, v_T^i, and w_T^i, $i \in \mathbb{Z}_3$. The three boundary curves of holes in the torus are the three 3-cycles $C_T^i = (u_T^i v_T^i w_T^i)$, $i \in \mathbb{Z}_3$. We assume that the torus is disjoint from all surfaces S^i and that its orientation induces the clockwise cyclic permutations $(u_T^i w_T^i v_T^i)$ of vertices on the boundary curves C_T^i, respectively. Notice the important difference between the cyclic permutations (uvw) on S and $(u_T^i w_T^i v_T^i)$ on the torus.

Now, for each $i \in \mathbb{Z}_3$, remove from the embedding ϕ^i the open triangular face $T^i = f^i(T)$. We thus create a new hole in each S^i with boundary curve C^i where $C^i = f^i(C)$ is the 3-cycle $(u^i v^i w^i)$ in ϕ^i. Finally, for $i \in \mathbb{Z}_3$ we identify the closed curve C^i in the embedding ϕ^i with the curve C_T^i in the embedding ψ_T in such a

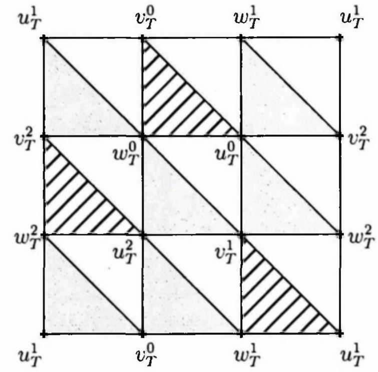


Figure 3.3: The toroidal embedding ψ_T of $K_{3,3,3}$.

way that $u^i \equiv u_T^i$, $v^i \equiv v_T^i$, and $w^i \equiv w_T^i$. Applying this procedure successively to each white triangular face $T \in \mathcal{T}$ and assuming that the corresponding auxiliary toroidal embeddings ψ_T are mutually disjoint, we obtain from S^0, S^1, and S^2 a new connected triangulated surface with boundary, which we denote by $\hat{S}$. Roughly speaking, $\hat{S}$ is obtained from S^0, S^1, and S^2 by adding $|T|$ toroidal "bridges" raised, for each $T \in \mathcal{T}$, above the white triangular faces $T^i = f^i(T)$, $i \in \mathbb{Z}_3$.

Clearly, $\hat{S}$ has six holes, and their disjoint boundary curves correspond to the cycles $D_k^i = f^i(D_k)$ in the graphs G^i, $i \in \mathbb{Z}_3$ $k = 0, 1$. Also, it is easy to see that the chosen orientations of ϕ^i and ψ_T guarantee that the bordered surface $\hat{S}$ is orientable. In fact, $\hat{S}$ inherits the clockwise orientation from the embeddings ϕ^i, $i \in \mathbb{Z}_3$, and ψ_T, $T \in \mathcal{T}$. Note that $\hat{S}$ also inherits the proper two-colouring of triangular faces from these embeddings. Since we have $t = (n-1)(n-3)/6$ black triangles in S, and hence in each S^i, and for each of the t white triangles T in S we added, in the auxiliary toroidal embedding ψ_T, another 15 triangles, the total number of triangular faces on $\hat{S}$ is equal to $3t + 15t = 3(n-1)(n-3)$. For each collection of 15 triangles added, 9 are white and 6 are black; hence it is easy to check that exactly half of the triangles on $\hat{S}$ are black, as expected.

Let H be the graph that triangulates the bordered surface $\hat{S}$; we need a precise description of H. Let $D_0 = (u_1 u_2 \cdots u_{(n-1)/2})$ and $D_1 = (v_1 v_2 \cdots v_{(n-1)/2})$ be our disjoint cycles in $G \simeq K_{n-1}$. Since n is odd, every other edge of D_0 and D_1 is incident to a white triangle on $\hat{S}$; let these edges be $u_2u_3, u_4u_5, \ldots, u_{(n-1)/2}u_1$ and

$v_2v_3, v_4v_5, \ldots, v_{(n-1)/2}v_1$ respectively. From the above construction it is easy to see that the graph H is obtained as follows. For $1 \leq j \neq j' \leq (n-1)/2$, each vertex u_j and v_j of G gives rise to three vertices u_j^0, u_j^1, u_j^2 and v_j^0, v_j^1, v_j^2 of H, each edge $u_ju_{j'}$ and $v_jv_{j'}$ of G incident to a black triangle gives rise to 3 edges $u_j^iu_{j'}^i$ and $v_j^iv_{j'}^i$, and finally each edge $u_ju_{j'}$ and $v_jv_{j'}$ of G incident to a white triangle gives rise to 9 edges $u_j^iu_{j'}^{i'}$ and $v_j^iv_{j'}^{i'}$, $i, i' \in \mathbb{Z}_3$, of H. Since each edge of G, apart from the $(n-1)/2$ edges $u_1u_2, u_3u_4, \ldots, u_{(n-3)/2}u_{(n-1)/2}$ and $v_1v_2, v_3v_4, \ldots, v_{(n-3)/2}v_{(n-1)/2}$, is incident to exactly one white triangle, the total number of edges of the graph H is $9(|E(G)| - (n-1)/2) + 3(n-1)/2 = 3(n-1)(3n-8)/2$.

For each edge $u_ju_{j'}$ and $v_jv_{j'}$ of $G \simeq K_{n-1}$, H contains all edges of the form $u_j^iu_{j'}^{i'}$ and $v_j^iv_{j'}^{i'}$, $i, i' \in \mathbb{Z}_3$ except when $\{u_j, u_{j'}\} = \{u_l, u_{l+1}\}$ and $\{v_j, v_{j'}\} = \{v_l, v_{l+1}\}, l = 1, 3, 5, \ldots, (n-3)/2$. Specifically, if $\{u_j, u_{j'}\} = \{u_l, u_{l+1}\}$ and $\{v_j, v_{j'}\} = \{v_l, v_{l+1}\}$, $l = 1, 3, 5, \ldots, (n-3)/2$ then H contains no edges of the form $u_j^iu_{j'}^{i'}$ and $v_j^iv_{j'}^{i'}$ with $i \neq i'$. Also, H contains no edge of the form $u_j^iu_j^{i'}$ and $v_j^iv_j^{i'}$, $i, i' \in \mathbb{Z}_3$. We see that, abstractly, H is isomorphic to K_{3n-3} minus $(n-1)$ pairwise disjoint 3-cycles of the form $(u_j^0u_j^1u_j^2)$ and $(v_j^0v_j^1v_j^2)$, $1 \leq j \leq (n-1)/2$ and minus $(n-1)/2$ pairwise disjoint 6-cycles $(u_l^0u_{l+1}^1u_l^2u_{l+1}^0u_l^1u_{l+1}^2)$ and $(v_l^0v_{l+1}^1v_l^2v_{l+1}^0v_l^1v_{l+1}^2)$, $l = 1, 3, 5, \ldots, (n-3)/2$.

Let $\omega : H \to \hat{S}$ be the embedding constructed above. We recall that the boundary curves of the six holes in $\hat{S}$ are D_0^i and D_1^i, the images of the cycles D_0 and D_1 respectively under the isomorphisms f_i, $i \in \mathbb{Z}_3$. In order to complete the construction and obtain a face two-colourable triangular embedding of K_{3n-2} we need one more modification of the bordered surface $\hat{S}$. We build two more auxiliary triangulated bordered surfaces $\bar{S}_0$ and $\bar{S}_1$ and paste them to $\hat{S}$ so that all six holes of $\hat{S}$ will be capped. The bordered surfaces $\bar{S}_0$ and $\bar{S}_1$ should contain the edges which are missing from $\hat{S}$ and also the edge sides, white or black, which are on the border $\hat{S}$; edges of the form $u_j^iu_j^{i'}$, $v_j^iv_j^{i'}$, $i \neq i'$ and $u_l^iu_{l+1}^{i'}$, $v_l^iv_{l+1}^{i'}$, $i, i' =\in \mathbb{Z}_3$, white edge sides of the form $u_l^iu_{l+1}^i$, $v_l^iv_{l+1}^i$ and black edge sides of the form $u_{l+1}^iu_{l+2}^i$, $v_{l+1}^iv_{l+2}^i$, $l = 1, 3, 5, \ldots, (n-3)/2$.

Let λ_0 and λ_1 be the toroidal embeddings of the multigraphs L_0 and L_1 respectively with faces of length 1 and 3 coloured black and white, as is L_0 depicted in Figure 3.4; L_1 can be obtained by the mapping $u_j \to v_j$, $1 \le j \le (n-1)/2$. Our Figure 3.4 also shows voltages α on directed edges of L_0, taken in the group $\mathbb{Z}_3 = \{0,1,2\}$. We deliberately use the same letters for vertices of L_0 and L_1 as for vertices of the graphs G_i but assume that these graphs are disjoint; such notation will be of advantage later. The lifted graphs L_0^α and L_1^α have the vertex set $\{u_j^i; 1 \le j \le (n-1)/2, i \in \mathbb{Z}_3\}$ and $\{v_j^i; 1 \le j \le (n-1)/2, i \in \mathbb{Z}_3\}$ respectively. The edge set of L_0^α and L_1^α can be described as follows. For each fixed $l = 1,3,5,\ldots,(n-3)/2$, the 6 vertices u_l^i, u_{l+1}^i and the 6 vertices v_l^i, v_{l+1}^i, $i \in \mathbb{Z}_3$, induce the complete graphs $J_l \simeq K_6$ and $J_l' \simeq K_6$ in L_0^α and L_1^α respectively. Furthermore, two successive complete subgraphs J_l and J_{l+2} and J_l' and J_{l+2}' are joined by the three edges $u_{l+1}^i u_{l+2}^i$ and $v_{l+1}^i v_{l+2}^i$, $i \in \mathbb{Z}_3$ respectively. Thus we have a total of $15(n-1)/4 + 3(n-1)/4 = 9(n-1)/2$ edges in each lifted graph L_0^α and L_1^α and there are neither loops nor multiple edges there.

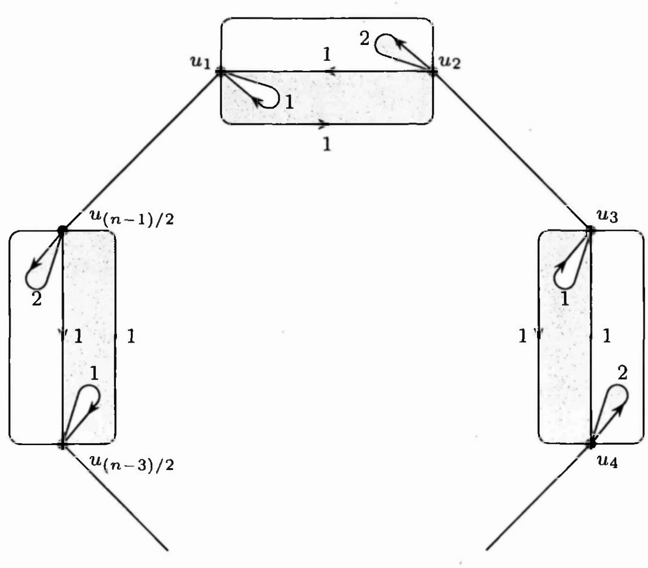


Figure 3.4: The plane embedding of the multigraph L_0.

The lifted embeddings $\lambda_k^\alpha : L_k^\alpha \to \bar{S}_k$, $k = 0,1$ have $2(n-1)$ triangular faces each. The white ones are bounded by the triangles $(u_l^0 u_l^1 u_l^2)$, $(u_l^0 u_{l+1}^0 u_{l+1}^2)$,

$(u_l^1 u_{l+1}^1 u_{l+1}^0)$, $(u_l^2 u_{l+1}^2 u_{l+1}^1)$ and $(v_l^0 v_l^1 v_l^2)$, $(v_l^0 v_{l+1}^0 v_{l+1}^2)$, $(v_l^1 v_{l+1}^1 v_{l+1}^0)$, $(v_l^2 v_{l+1}^2 v_{l+1}^1)$. The black ones are bounded by $(u_{l+1}^0 u_{l+1}^2 u_{l+1}^1)$, $(u_{l+1}^0 u_l^1 u_l^2)$, $(u_{l+1}^1 u_l^2 u_l^0)$, $(u_{l+1}^2 u_l^0 u_l^1)$ and $(v_{l+1}^0 v_{l+1}^2 v_{l+1}^1)$, $(v_{l+1}^0 v_l^1 v_l^2)$, $(v_{l+1}^1 v_l^2 v_l^0)$, $(v_{l+1}^2 v_l^0 v_l^1)$, where $l = 1, 3, \ldots, (n-3)/2$. In addition, there are four more faces in each embedding; three faces, which we denote by F_0^i and F_1^i, bounded by $((n-1)/2)$-gons of the form $(u_1^i u_2^i \cdots u_{(n-1)/2}^i)$ and $(v_1^i v_2^i \cdots v_{(n-1)/2}^i)$, $i \in \mathbb{Z}_3$, and one face F_0' and F_1' bounded by the $((3n-3)/2)$-gon $(u_1^0\, u_2^1\, u_3^1\, u_4^2 \cdots\ u_{(n-3)/2}^2\, u_{(n-1)/2}^0)$ and $(v_1^0\, v_2^1\, v_3^1\, v_4^2 \cdots\ v_{(n-3)/2}^2\, v_{(n-1)/2}^0)$.

Let us now cut out from each $\bar{S}_k$ the three open faces F_k^i, $i \in \mathbb{Z}_3$, bounded by the above three disjoint $(n-1)/2$-gons, obtaining thereby two orientable bordered surfaces S_k^*, $k = 0, 1$. Let L_k^* be the graphs obtained from L_k^α by adding the same new vertex ∞^* in each graph and joining it to each vertex of L_k^α while keeping all edges in L_k^α unchanged. We construct the embeddings $\lambda_k^* : L_k^* \to S_k^*$ from λ_k^α in an obvious way. In the embeddings λ_k^α, after the removal of the three open faces, we insert the vertex ∞^* in the centre of each face F_k' bounded by the $((3n-3)/2)$-gon and join this point by open arcs, within each F_k', to every vertex on the boundary of F_k'. By doing this, instead of F_k' we now have $(3n-3)/2$ new triangular faces on each S_k^*; they are bounded by the 3-cycles $\infty^* u_j^i u_{j+1}^{i'}$ and $\infty^* v_j^i v_{j+1}^{i'}$, $i, i' \in \mathbb{Z}_3$. We now colour the new triangular faces as follows: the face of λ_0^* and λ_1^* bounded by the 3-cycle $\infty^* u_j^i u_{j+1}^{i'}$ and $\infty^* v_j^i v_{j+1}^{i'}$ respectively will be black (white) if the triangular face of the embedding λ_0^α and λ_1^α containing the edge $u_j^i u_{j+1}^{i'}$ and $v_j^i v_{j+1}^{i'}$ respectively is white (black). It is easy to check that this rule indeed well defines a two-colouring of the triangular embeddings $\lambda_k^* : L_k^* \to S_k^*$, $k = 0, 1$. We thus have $2(n-1) + (3n-3)/2 = 7(n-1)/2$ triangular faces on each S_k^*, a total of $7(n-1)$, exactly half of which are black.

We are ready for the final step of the construction. Our method of constructing the orientable surface $\hat{S}$ guarantees that a chosen orientation of $\hat{S}$ induces consistent orientations of the boundary cycles of the six holes of $\hat{S}$; we may assume that the orientation induces the cyclic ordering of the cycles D_0^i and D_1^i in the form that was used before, namely, $D_0^i = f^i(D_0) = (u_1^i u_2^i \cdots u_{(n-1)/2}^i)$ and

$D_1^i = f^i(D_1) = (v_1^i v_2^i \cdots v_{(n-1)/2}^i)$, $i \in \mathbb{Z}_3$. The bordered surfaces S_0^* and S_1^* have three holes each, and again, the method of construction implies that an orientation of each S_0^* and S_1^* can be chosen so that the boundary cycles of the holes are oriented in the form $D_0^{*i} = (u_{(n-1)/2}^i \cdots u_2^i u_1^i)$ and $D_1^{*i} = (v_{(n-1)/2}^i \cdots v_2^i v_1^i)$, $i \in \mathbb{Z}_3$. It remains to do the obvious, that is, paste together the boundary cycles D_0^i, D_1^i and D_0^{*i}, D_1^{*i} so that corresponding vertices u_j^i and v_j^i get identified and furthermore to identify the vertex ∞^* from S_0^* and S_1^*. As a result, we obtain an orientable pseudosurface $\hat{S}\#S^*$ with one regular pinch point of multiplicity 2, known as the connected sum of the bordered surfaces $\hat{S}$ and S^*, and a triangular embedding $\sigma : K \to \hat{S}\#S^*$ of some graph K. We claim that $K \simeq K_{3n-2}$ and that the triangulation is face two-colourable.

Obviously, $|V(K)| = 3n - 2$. An edge count shows that $|E(K)| = |E(H)| + 2|E(L_0^*)| - 6|E(D)| = 3(n-1)(3n-8)/2 + 12(n-1) - 3(n-1) = (3n-2)(3n-3)/2$. It is easy to check that, except for edges incident with ∞^* and edges contained in the six $(n-1)/2$-cycles D_k^{i*}, the graphs L_k^* contain exactly those edges which are missing in H. This shows that there are no repeated edges or loops in K, and thus $K \simeq K_{3n-2}$. As far as the face-two-colouring is concerned, we just have to see what happens along the identified $(n-1)/2$-cycles D_k^i and D_k^{i*}, $k = 0, 1$, since both triangulations of $\hat{S}$ and S_k^* are already known to be face two-colourable. But according to the construction, if $l = 1, 3, 5, \ldots, (n-3)/2$, a triangular face on $\hat{S}$ that contains the edge $u_l^i u_{l+1}^i$ or $v_l^i v_{l+1}^i$ is black, while the face on S_0^* and S_1^* bounded by the triangle $(u_l^i u_{l+1}^i u_{l+1}^{i-1})$ and $(v_l^i v_{l+1}^i v_{l+1}^{i-1})$ is white. ■

As a consequence of the above theorem, we have the following corollary which gives an infinite class of orientable pseudosurface embeddings.

Corollary 3.1.2 *For $n = 3^s \cdot 12 + 1$, $s \geq 0$, the complete graph K_n has a face two-colourable triangular embedding on a pseudosurface with one regular pinch point of multiplicity 2.*

3.2 General construction

In this section we present the main result of this chapter. We give a construction which is a generalization of the construction given above. This construction is a modification of the product construction described as Construction 4 of [29]. As a corollary of this more general construction we obtain infinite linear classes of orientable pseudosurface embeddings as stated at the beginning of this chapter.

Theorem 3.2.1 *Suppose that* $n \equiv 1 \pmod{12}$ *and that* $m \equiv 1$ *or* $3 \pmod 6$. *Then if there exists a face two-colourable triangular embedding of the complete graph* K_n *in an orientable pseudosurface having precisely one regular pinch point of multiplicity 2, then there exists a face two-colourable triangular embedding of the complete graph* $K_{m(n-1)+1}$ *in an orientable pseudosurface having precisely one regular pinch point of multiplicity 2.*

Proof To facilitate a comparison of the steps carried out here with the original proof we will keep to the notation of [29] as much as possible. A rough outline of the proof is as follows. We will begin by taking m copies of a face two-colourable triangulation of K_n in an orientable surface with one regular pinch point of multiplicity 2, removing the m pinch points together with their incident edges and faces, and 'bridging' the m components in an intricate way to obtain a connected surface with $2m$ cyclic boundary components. We will continue by capping the $2m$ 'holes' created in the previous step by a cap consisting of a bordered pseudosurface with one pinch point and $2m$ cyclic boundary components. We will show that from this construction we obtain a face two-colourable triangulation of $K_{m(n-1)+1}$, or equivalently a pair of STS$(m(n-1)+1)$s, in an orientable surface with one regular pinch point of multiplicity 2.

Let η be a face two-colourable orientable triangulation of K_n in a (say, clockwise) oriented pseudosurface with a single regular pinch point of multiplicity 2, with faces properly coloured black and white. Let z be the unique vertex of K_n identified with the pinch point. We remove from η the vertex z, together with all open arcs and open triangular faces originally incident with z, obtaining a face two-coloured triangular embedding ϕ of $G = K_n \setminus \{z\} \cong K_{n-1}$ in a bordered surface S. Observe that S has no pinch points and the two connected boundary components of S are two disjoint cycles D_1 and D_2 in G, each of length $(n-1)/2$. Following our outline, for every $i \in \mathbb{Z}_m$ let $\phi^i : G^i \to S^i$ be m mutually disjoint copies of the embedding ϕ together with the proper two-colouring of triangular faces inherited from η. In doing so we assume that the natural mapping $f^i : G \to G^i$ that endows each vertex of G with the superscript i is a colour-preserving and orientation-preserving isomorphism of the embeddings ϕ and ϕ^i. Initially we will assume that m and $(n-1)/2$ are relatively prime, and we will deal with the general case at the end of the proof.

We continue with describing the 'bridging' procedure. To do so we need to return to the embedding ϕ whose description uses no superscripts. Let $\mathcal{T}$ be the set of the total of $t = (n-1)(n-3)/6$ white triangular faces in ϕ and for each $i \in \mathbb{Z}_m$ let $\mathcal{T}^i = f^i(\mathcal{T})$ be the corresponding set of all white triangular faces in ϕ^i. Choose a particular triangular face T of ϕ with vertex set $\{a, b, c\}$ and assume that the cyclic permutation (abc) corresponds to the clockwise orientation of the boundary cycle C of T. For each such T take a face two-colourable orientable triangulation ψ_T of the complete tripartite graph $K_{m,m,m}$ in a closed surface S_T disjoint from each S^i; let $\{a^i_T\}, \{b^i_T\}$ and $\{c^i_T\}$, $i \in \mathbb{Z}_m$, be the three vertex-parts of this $K_{m,m,m}$. By Construction 1 of [29], we may select ψ_T to have a parallel class of black triangular faces $\{a^i_T, b^i_T, c^i_T\}$ and we may choose the orientation of ψ_T to ensure that it induces the cyclic permutations $(a^i_T c^i_T b^i_T)$ of the boundary cycles C^i_T of these faces. Note that we have chosen different cyclic permutations (abc) on S and $(a^i_T c^i_T b^i_T)$ on S_T.

Next, for every $i \in \mathbb{Z}_m$ we perform the following steps: remove from ϕ^i the open triangular face $T^i = f^i(T)$, creating in each S^i a new hole with boundary curve $C^i = f^i(C)$ corresponding to the 3-cycle $(a^i b^i c^i)$ in ϕ^i, remove from ψ_T the open triangular faces $\{a^i_T, b^i_T, c^i_T\}$, and identify the closed curve C^i in ϕ^i with the curve

C_T^i in ψ_T in such a way that $a^i \equiv a_T^i$, $b^i \equiv b_T^i$, and $c^i \equiv c_T^i$. Assuming that the embeddings ψ_T are mutually disjoint, we apply this procedure successively to *each* white triangular face $T \in \mathcal{T}$. Let $\hat{S}$ denote the connected triangulated surface $\hat{S}$ with $2m$ boundary components, obtained this way from the surfaces S^i. Roughly speaking, $\hat{S}$ is obtained from the surfaces S^i by adding $|\mathcal{T}|$ 'bridges', explaining the term 'bridging' used in the earlier informal outline of our construction.

The $2m$ boundary components of $\hat{S}$ correspond, for $i \in \mathbb{Z}_m$, to the cycles $D_1^i = f^i(D_1)$ and $D_2^i = f^i(D_2)$ in the graphs G^i, the images of the cycles D_1 and D_2 in G. The chosen orientations of ϕ^i and ψ_T induce an orientation of $\hat{S}$ by inheriting the clockwise orientation from ϕ^i and ψ_T, and $\hat{S}$ also inherits the proper two-colouring of triangular faces from these embeddings. Note that there are $t = (n-1)(n-3)/6$ black triangles in S, and hence in each S^i, and for each of the t white triangles T in S we added, in ψ_T, another $(2m^2 - m)$ triangles. The total number of triangular faces on $\hat{S}$ is therefore equal to $mt + (2m^2 - m)t = m^2(n-1)(n-3)/3$. For each collection of $(2m^2 - m)$ triangles added, m^2 are white and $(m^2 - m)$ are black; hence it is easy to check that exactly half of the triangles on $\hat{S}$ are black, as expected.

To proceed, we need an exact description of the graph H triangulating the bordered surface $\hat{S}$. Let $D_1 = (u_1u_2 \dots u_{(n-1)/2})$ and $D_2 = (v_1v_2 \dots v_{(n-1)/2})$ be the two cycles in $G = K_n \setminus \{z\}$ introduced earlier. Since $(n-1)/2$ is even, every other edge of both D_1 and D_2 is incident to a white triangle on $\hat{S}$; let these edges be u_2u_3, u_4u_5, $\dots$, $u_{(n-1)/2}u_1$ and v_2v_3, v_4v_5, $\dots$, $v_{(n-1)/2}v_1$. It may now be checked that the graph H is obtained as follows. For $1 \le j \ne j' \le (n-1)/2$, each vertex u_j and v_j of G gives rise to m vertices u_j^i and v_j^i, $i \in \mathbb{Z}_m$, of H, and each edge $u_ju_{j'}$ and $v_jv_{j'}$ of G incident to a white triangle gives rise to m^2 edges $u_j^iu_{j'}^{i'}$ and $v_j^iv_{j'}^{i'}$, $i, i' \in \mathbb{Z}_m$, of H. Since each edge of G except for the $(n-1)/2$ edges u_1u_2, u_3u_4, $\dots$, $u_{(n-3)/2}u_{(n-1)/2}$ and v_1v_2, v_3v_4, $\dots$, $v_{(n-3)/2}v_{(n-1)/2}$ is incident to exactly one white triangle, H has $m^2(|E(G)| - (n-1)/2) + m(n-1)/2 = m(n-1)(m(n-3)+1)/2$ edges. To have further insight into its structure, observe that for each edge $u_ju_{j'}$ and $v_jv_{j'}$ of $G \cong K_{n-1}$, except when $\{u_j, u_{j'}\} = \{u_l, u_{l+1}\}$ and $\{v_j, v_{j'}\} = \{v_l, v_{l+1}\}$, $l = 1, 3, 5, \ldots, (n-3)/2$, H contains all edges of the form $u_j^i u_{j'}^{i'}$ and $v_j^i v_{j'}^{i'}$, $i, i' \in \mathbb{Z}_m$. However, if $\{u_j, u_{j'}\} = \{u_l u_{l+1}\}$ or $\{v_j, v_{j'}\} = \{v_l v_{l+1}\}$ for some $l = 1, 3, \ldots, (n-3)/2$ then H contains no edge $u_j^i u_{j'}^{i'}$ and $v_j^i v_{j'}^{i'}$ with $i \neq i'$, although it does contain the edges $u_j^i u_{j'}^i$ and $v_j^i v_{j'}^i$. Note also that H contains no edges of the form $u_j^i u_j^{i'}$ and $v_j^i v_j^{i'}$ for any $i, i' \in \mathbb{Z}_m$. It follows that H is isomorphic to $K_{m(n-1)}$ minus $(n-1)/2$ pairwise disjoint copies of (K_{2m} minus a 1-factor), one on each of the sets $\{u_l^0, u_l^1, \ldots, u_l^{m-1}, u_{l+1}^0 u_{l+1}^1, \ldots, u_{l+1}^{m-1}\}$ and $\{v_l^0, v_l^1, \ldots, v_l^{m-1}, v_{l+1}^0 v_{l+1}^1, \ldots, v_{l+1}^{m-1}\}$ with the missing 1-factor $\{u_l^i u_{l+1}^i;\ i \in \mathbb{Z}_m\}$ and $\{v_l^i v_{l+1}^i;\ i \in \mathbb{Z}_m\}$, respectively, for $l = 1, 3, 5, \ldots, (n-3)/2$.

Let $\omega : H \to \hat{S}$ be the resulting embedding of H in our surface $\hat{S}$ with $2m$ boundary components consisting of the images of the cycles D_1 and D_2 under the isomorphisms f^i, $i \in \mathbb{Z}_m$. To construct the final face two-colourable orientable triangulation of $K_{m(n-1)+1}$ we build two auxiliary triangulated bordered surface S_1^* and S_2^* containing m boundary components each, and paste them to $\hat{S}$ so that the $2m$ holes of $\hat{S}$ will be capped. We will focus on S_1^* in detail and then explain how S_2^* is obtained. The surface S_1^* will be described as a lift of the plane embedding μ_1 of the multigraph M_1 as depicted in Figure 3.5, with voltages α on directed edges of M_1 in the group $\mathbb{Z}_m$ identical with the group from which all our superscripts are taken. Edges with no direction assigned are assumed to carry the zero voltage.

The lifted graph M_1^α has the vertex set $\{u_j^i : 1 \leq j \leq (n-1)/2, i \in \mathbb{Z}_m\}$. We are deliberately using the same letters for vertices of M^α as for vertices of the graphs G^i, but assume that these graphs are disjoint; such notation will be of advantage later. The lifted embedding $\mu_1^\alpha : M_1^\alpha \to R_1$ is orientable and has the following face boundaries.

(a) $(n-1)/4$ faces whose boundaries correspond to cycles of length $2m$ of the form $(u_{2j-1}^0 u_{2j}^0 u_{2j-1}^{m-1} u_{2j}^{m-1} \ldots u_{2j}^1)$ for $1 \leq j \leq (n-1)/4$.

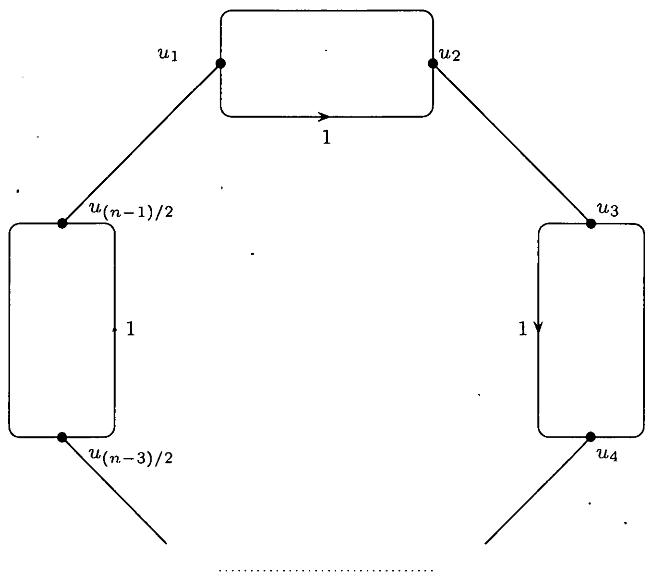


Figure 3.5: The plane embedding μ_1 of the graph M_1.

(b) m faces whose boundaries correspond to cycles of length $(n-1)/2$ of the form $(u^i_{(n-1)/2}u^i_{(n-3)/2}\dots u^i_1)$ for $i \in \mathbb{Z}_m$.

(c) One face whose boundary corresponds to a cycle of length $m(n-1)/2$ of the form $(u^0_1u^1_2u^1_3u^2_4u^2_5u^3_6\dots u^0_{(n-1)/2})$. (**Note:** This is the only place in this proof where we have used the assumption that m and $(n-1)/2$ are relatively prime; if this were not the case then a multiplicity of faces with shorter boundary cycles would be obtained.)

We now describe a series of modifications of the embedding μ_1. Firstly, we remove all the open faces of type (a) from the surface R_1, leaving an orientable surface R^o_1 with $(n-1)/4$ vertex-disjoint boundaries $(u^0_{2j-1}u^0_{2j}\ u^{m-1}_{2j-1}u^{m-1}_{2j}\dots u^1_{2j})$, $1 \le j \le (n-1)/4$. We cap each of these in turn by taking, for each j, a face two-colourable orientable triangulation of K_{2m+1} with colour classes black and white on the vertex set $\{\infty_j, u^0_{2j}, u^0_{2j-1}, u^1_{2j}, u^1_{2j-1}, \dots, u^{m-1}_{2j-1}\}$, in which the rotation at ∞_j is the cycle $(u^0_{2j}u^0_{2j-1}u^1_{2j}u^1_{2j-1}\dots u^{m-1}_{2j-1})$ and in which the face corresponding to the 3-cycle $(\infty_j u^0_{2j}u^0_{2j-1})$ is coloured black. Here also for convenience we are using the same letters for the vertices of our K_{2m+1} embeddings as for the vertices of M^α_1, but we assume that the corresponding surfaces are disjoint. Secondly, from each embedding of K_{2m+1} we remove the vertex ∞_j, all open edges incident with ∞_j, and all open triangular faces incident with ∞_j. This results in a face two-colourable embedding of K_{2m} in an orientable surface R_{1j} with a boundary cycle $(u^0_{2j}u^0_{2j-1}u^1_{2j}u^1_{2j-1}\dots u^{m-1}_{2j-1})$. Thirdly, for every j such that $1 \le j \le (n-1)/4$ we glue the surface R_{1j} to the surface R^o_1, identifying points carrying the same labels on each of the two surfaces, thereby obtaining an embedding $\mu'_1 : M'_1 \to R'_1$ of a graph M'_1 with $m^2(n-1)/2$ edges.

We continue by removing from R'_1 all the open faces of type (b), obtaining thus an orientable surface S^*_1 with m vertex-disjoint boundaries of the form $(u^i_{(n-1)/2}u^i_{(n-3)/2}\dots u^i_1)$, $i \in \mathbb{Z}_m$.

Let M^*_1 be the graph obtained from M'_1 by adding a new vertex $\infty_{(1)}$ and joining it to each vertex of M'_1 while keeping all other edges in M'_1 unchanged. We construct an embedding $\mu^*_1 : M^*_1 \to S^*_1$ from the embedding of M'_1 in S^*_1 by inserting the vertex $\infty_{(1)}$ in the centre of the face F_1 bounded by the cycle of length $m(n-1)/2$ and joining this vertex by open arcs within F_1 to *every* vertex on the boundary of F_1 (that is, to every vertex of M^α_1). This gives rise to $m(n-1)/2$ new triangular faces on S^*_1 bounded, for $1 \le j \le (n-1)/4$, by cycles of the forms $(\infty_{(1)}u^k_j u^{k+1}_{j+1})$ for j odd, and $(\infty_{(1)}u^k_j u^k_{j+1})$ for j even. The new triangular faces will be coloured as described in the paragraph that follows.

The edge $u^0_1u^1_2$ lies in a black triangular face of μ'_1 because $(\infty_1 u^0_1 u^1_2)$ was a white triangular face of the K_{2m+1} embedding employed in the construction of μ'_1. We therefore colour white the face of μ^*_1 bounded by the 3-cycle $(\infty_{(1)}u^0_1u^1_2)$. It is easy to see that, by an extension of this argument, we must colour white those alternate triangles with boundary cycles $(\infty_{(1)}u^k_j u^{k+1}_{j+1})$ for j odd. The remaining alternate triangles, those with boundary cycles of the form $(\infty_{(1)}u^k_j u^k_{j+1})$ for j even, do not share an edge with any existing triangular face of μ'_1 and these are coloured black.

As a result of this process, the triangular faces of μ^*_1 are properly two-coloured, and the number of such faces is

$$\frac{(n-1)}{4}\frac{2m(2m-2)}{3}+\frac{m(n-1)}{2}=\frac{m(2m+1)(n-1)}{6},$$

where the terms $(n-1)/4$, $2m(2m-2)/3$ and $m(n-1)/2$ on the left represent the number of faces of type (a) in R_1, the number of triangles in the added K_{2m} and the number of triangles added by inserting the vertex $\infty_{(1)}$, respectively. Note that exactly half of these faces are coloured black.

The next step is to construct an embedding μ_2^* of a graph M_2^* on a surface S_2^* with the extra vertex $\infty_{(2)}$, which is done in exactly the same way as described above for μ_1^* by replacing all occurrences of u with v, keeping all subscripts and superscripts unchanged. The description would thus start from an embedding μ_2 of a graph M_2 with vertices $v_1, v_2, \ldots, v_{(n-1)/2}$ corresponding to Figure 3.5 and continue through the intermediate graphs, surfaces and embeddings M_2^α, R_2, μ_2^α, R_2^o, M_2', R_2', μ_2' to M_2^*, S_2^* and μ_2^* as indicated. The embedding μ_2^* will, of course, have the same number of triangles as μ_1^* given above, half of which will be black.

We are ready for the final steps. Our method of constructing the orientable surface $\hat{S}$ from the earlier part of the proof guarantees that a chosen orientation of $\hat{S}$ induces *consistent* orientations of the boundary cycles of the $2m$ holes of $\hat{S}$. We may assume that the orientation induces the cyclic ordering of the cycles D_1^i and D_2^i in the form that was used before, namely, $D_1^i = f^i(D_1) = (u_1^i u_2^i \ldots u_{(n-1)/2}^i)$ and $D_2^i = f^i(D_2) = (v_1^i v_2^i \ldots v_{(n-1)/2}^i)$, $i \in \mathbb{Z}_m$. The bordered surfaces S_1^* and S_2^* have m holes each. Our construction again implies that an orientation of S_1^* and S_2^* can be chosen so that the boundary cycles are oriented in the form $D_1^{i*} = (u_{(n-1)/2}^i \ldots u_2^i u_1^i)$ and $D_2^{i*} = (v_{(n-1)/2}^i \ldots v_2^i v_1^i)$, $i \in \mathbb{Z}_m$. It remains to do the obvious, namely, for each i to paste together the boundary cycles D_1^i and D_1^{i*} in such a way that the corresponding vertices u_j^i get identified, and glue the boundary cycles D_2^i and D_2^{i*} so that the corresponding vertices v_j^i will be identified. Finally, we identify the vertex $\infty_{(1)}$ with $\infty_{(2)}$, creating one regular pinch point of multiplicity 2.

The final result is an orientable pseudosurface $\bar{S}$ with a single pinch point, regular of multiplicity 2, and a triangular embedding $\sigma : K \to \bar{S}$ of some graph K. We claim that $K \cong K_{m(n-1)+1}$ and that the triangulation is face two-colourable. Obviously, $|V(K)| = m(n-1)+1$. A straightforward edge count shows that

$$\begin{aligned} |E(K)| &= |E(H)| + |E(M_1^*)| + |E(M_2^*)| - m|E(D_1)| - m|E(D_2)| \\ &= \frac{m(n-1)(m(n-3)+1)}{2} + (n-1)(m^2+m) - m(n-1) \\ &= \frac{m(n-1)(m(n-1)+1)}{2} = |E(K_{m(n-1)+1})|. \end{aligned}$$

It is easy to verify that, except for edges incident with the vertex obtained by identification of $\infty_{(1)}$ with $\infty_{(2)}$ and edges contained in the $2m$ cycles D_1^{i*} and D_2^{i*} of length $(n-1)/2$, the graph $M_1^* \cup M_2^*$ contains *exactly those edges which are missing in* H. This shows that there are no repeated edges or loops in K, and thus $K \cong K_{m(n-1)+1}$. As regards the face two-colouring, we just have to see what happens along the identified cycles D_1^i and D_1^{i*}, and D_2^i and D_2^{i*}, since the triangulations of $\hat{S}$, S_1^* and S_2^* have been face two-coloured. But according to the construction, if $l = 1, 3, 5, \ldots, (n-3)/2$, a triangular face on $\hat{S}$ that contains the edge $u_l^i u_{l+1}^i$ is black, while the face on S_1^* containing this edge is white because the embeddings of K_{2m+1} employed had the faces with boundary cycles $(\infty_j u_{2j}^i u_{2j-1}^i)$ coloured black. This also applies to the way the embeddings $\hat{S}$ and S_2^* meet.

To finish the proof it remains to deal with the case when m and $(n-1)/2$ are not relatively prime. To do so we return to Figure 3.5 and generalise the construction. Namely, it turns out that the voltages shown in Figure 3.5 as 1 may be replaced respectively by voltages $x_1, x_2, \ldots, x_{(n-1)/4} \in \mathbb{Z}_m$ provided that

(d) each x_i is relatively prime to m, and

(e) $\sum_{i=1}^{(n-1)/4} x_i$ is relatively prime to m.

Condition (d) ensures that the embedding μ_1^α will have $(n-1)/4$ faces with boundary cycles of length $2m$ on each of the sets of points of the form $\{u_{2j-1}^0, u_{2j}^0, u_{2j-1}^1, u_{2j}^1, \ldots, u_{2j-1}^{m-1}, u_{2j}^{m-1}\}$, while condition (e) ensures that μ_1^α has a single face with boundary cycle of length $m(n-1)/2$. In effect, condition (e) replaces the condition that m and $(n-1)/2$ should be relatively prime. Of course, a similar conclusion applies to the embedding μ_2^α. It is easy to see that there are numerous ways to select the voltages so that $x_j \in \{+1, -1\}$, $1 \leq j \leq (n-1)/4$, with $\sum_{i=1}^{(n-1)/4} x_i \in \{1, 2\}$, which is relatively prime to m since m is odd. One of these ways is to put $x_1 = 1$, $x_j = 1$ if j is even and $x_j = -1$ if j is odd and greater than 1. The subsequent steps in the proof then proceed as before with the obvious changes. ■

We now have the following corollary.

Corollary 3.2.2 *For all $n \equiv 13$ or $37 \pmod{72}$, there exists a biembedding of a pair of Steiner triple systems of order n in an orientable pseudosurface having precisely one regular pinch point of multiplicity 2.*

Proof Put $n = 13$ in the above theorem and use one of the biembeddings given in Section 3.1. ■

Remark The existence of such a biembedding of a pair of STS(25)s would extend the existence spectrum to include all $n \equiv 25$ or $73 \pmod{144}$, i.e. in arithmetic set density terms from 1/3 to 1/2 in the set of all $n \equiv 1 \pmod{12}$. We have tried to construct such a biembedding but have been unsuccessful.

CHAPTER 4

Triple systems of order 9

In this chapter, we consider some topological properties of the twofold triple systems of order 9, TTS(9)s. Up to isomorphism there are precisely 36 of these, which were enumerated in [45, 48]. These are listed in Appendix A, see page 63 of [8], and it is to this listing that we refer to throughout this chapter. Of these, numbers 1 to 23 contain repeated blocks and 24 to 36 are simple, i.e. contain no repeated blocks.

By sewing together the triples of a TTS(9) along common edges, a topological space is obtained which may be a surface, pseudosurface or generalised pseudosurface. A *generalised pseudosurface* is the connected topological space which results when finitely many identifications of finitely many points each, are made on a topological space of finitely many components each of which is a surface or a pseudosurface. The rotation schemes for these embeddings are listed in Appendix B.

In the next section we describe the structure of these 36 topological spaces. Then, in the final section we turn our attention to Steiner triple systems of order 9 and in particular to sets of these which are disjoint, again describing the topological properties of these sets.

4.1 TTS(9) embeddings

We first consider the 23 TTS(9)s which have repeated blocks. Clearly, when the triples are sewn together, these will form generalised pseudosurfaces. The reason is because separation at each point of a repeated block $\{a, b, c\}$ will yield a sphere. The table below lists some of the topological properties of each embedding. We consider the structure obtained when the generalised pseudosurfaces are separated at appropriate pinch points to form surfaces or pseudosurfaces. Orientable and nonorientable surfaces are denoted by S_g and N_γ respectively where g and γ are the orientable and nonorientable genus. The symbols S'_g and N'_γ denote pseudosurfaces, which are obtained when certain points of the surfaces S_g and N_γ respectively are identified to form pinch points. Additionally, p_i denotes a pinch point of multiplicity i.

	Number of pinch points	Structure of generalised pseudosurface	Face two-colourable
1	$9 \times p_4$	$12 \times S_0$	$\surd$
2	$2 \times p_2, 6 \times p_3, 1 \times p_4$	$7 \times S_0$	$\surd$
3	$3 \times p_2, 3 \times p_3$	$3 \times S_0, 1 \times N_2$	$\surd$
4	$8 \times p_2, 1 \times p_4$	$4 \times S_0, 1 \times S_1$	$\surd$
5	$6 \times p_2, 2 \times p_3, 1 \times p_4$	$4 \times S_0, 1 \times S'_0$	·
6	$6 \times p_2, 1 \times p_3$	$2 \times S_0, 1 \times N'_1$	$\surd$
7	$3 \times p_2, 2 \times p_3$	$2 \times S_0, 1 \times S'_1$	·
8	$6 \times p_2, 1 \times p_3$	$2 \times S_0, 1 \times N'_3$	·
9	$6 \times p_2, 3 \times p_3$	$4 \times S_0, 1 \times N_1$	·
10	$4 \times p_2, 4 \times p_3, 1 \times p_4$	$6 \times S_0$	·
11	$3 \times p_2, 2 \times p_3$	$2 \times S_0, 1 \times S'_1$	·
12	$5 \times p_2, 2 \times p_3$	$2 \times S_0, 1 \times S'_0$	·
13	$2 \times p_2, 3 \times p_3$	$2 \times S_0, 1 \times N'_1$	·
14	$4 \times p_2, 3 \times p_3$	$3 \times S_0, 1 \times N_1$	·

15	$3 \times p_2, 1 \times p_3$	$1 \times S_0, 1 \times N_2'$	·
16	$4 \times p_2$	$1 \times S_0, 1 \times N_3'$	·
17	$5 \times p_2$	$1 \times S_0, 1 \times N_2'$	·
18	$5 \times p_2$	$1 \times S_0, 1 \times N_2'$	·
19	$3 \times p_2$	$1 \times S_0, 1 \times N_4$	$\surd$
20	$5 \times p_2$	$1 \times S_0, 1 \times S_1'$	·
21	$9 \times p_2$	$3 \times S_0, 1 \times S_1$	$\surd$
22	$6 \times p_2$	$1 \times S_0, 1 \times N_1'$	·
23	$5 \times p_2, 1 \times p_3$	$2 \times S_0, 1 \times S_1$	·

Next, we consider the last 13 TTS(9)s which are simple. Since these systems are simple their embeddings do not necessarily form generalised pseudosurfaces. In the table below, G, P and S refer to generalised pseudosurface, pseudosurface and surface respectively.

	Type of surface	Number of pinch points	Structure of surface	Face two-colourable
24	G	$9 \times p_2$	$3 \times S_0$	·
25	G	$5 \times p_2$	$1 \times S_0, 1 \times N_2'$	·
26	P	$4 \times p_2$	$1 \times N_1'$	·
27	P	$2 \times p_2$	$1 \times N_3'$	·
28	P	$1 \times p_2$	$1 \times N_4'$	·
29	P	$4 \times p_2$	$1 \times N_1'$	·
30	P	$3 \times p_2$	$1 \times S_1'$	·
31	P	$3 \times p_2$	$1 \times S_1'$	·
32	P	$2 \times p_2$	$1 \times N_3'$	·
33	P	$3 \times p_2$	$1 \times S_1'$	$\surd$
34	P	$1 \times p_2$	$1 \times S_2'$	·
35	S	·	$1 \times N_5$	·
36	S	·	$1 \times N_5$	$\surd$

The most interesting of the above embeddings are probably numbers 30, 31, 33, 34, 35 and 36 which are either surfaces or orientable pseudosurfaces.

TTS(9) #35 is embedded in the nonorientable surface N_5. It is not face two-colourable and has the block set {012, 018, 023, 034, 045, 056, 067, 078, 124, 136, 137, 146, 157, 158, 238, 245, 257, 267, 268, 347, 356, 358, 468, 478}. The automorphism group is C_6 of order 6 and is generated by the permutation (0 3 6 2 1 7)(4 5 8). TTS(9) #36 is also embedded in the nonorientable surface N_5. However, it is face two-colourable and gives the unique surface biembedding of a pair of STS(9)s. A realization is obtained by taking the system with block set {012, 034, 056, 078, 136, 147, 158, 238, 245, 267, 357, 468} and the other obtained from this by applying the permutation $\theta = (0\ 2)(1\ 3)(6\ 7)(4)(5)(8)$. It has automorphism group $C_3 \times S_3$ of order 18 which, in this realization, is generated by the permutations θ and (0 1 8)(2 5 7)(3 4 6). The automorphisms of even order exchange the colour classes. These embeddings were found by Altshuler and Brehm [1], from which the given realizations and automorphism groups are taken, and rediscovered later by Bracho and Strausz [5].

Kramer and Mesner showed in [38] that there are two nonisomorphic pairs of disjoint STS(9)s. As such, there is one other face two-colourable embedding of a TTS(9). This is #33 and it is embedded in a torus with three regular pinch points of multiplicity 2 and seems to have been discovered by Emch [18]. The embedding is illustrated in Figure 4.1 below. We will refer to it as the Emch surface.

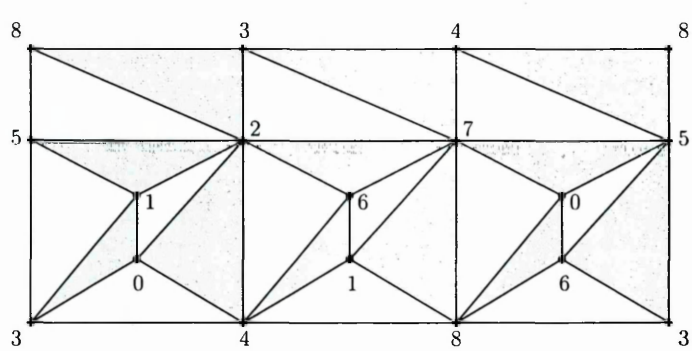


Figure 4.1: The Emch surface.

However, systems #30 and #31 also embed in a torus with 3 pinch points, though of course these are not face two-colourable. These embeddings do not seem to have appeared previously in the literature and are illustrated below in Figure 4.2 and Figure 4.3 respectively. Finally, system #34 embeds in a double torus with one pinch point and it is illustrated in Figure 4.4.

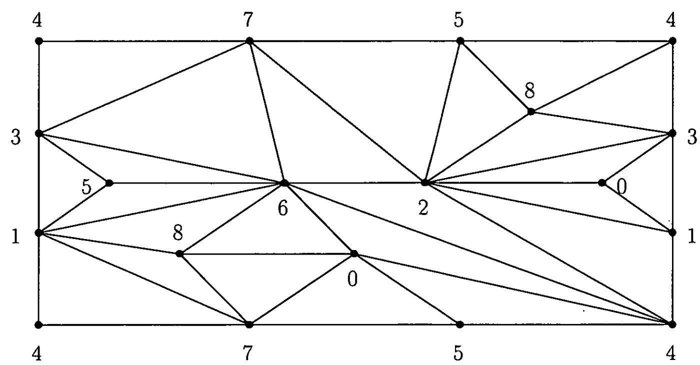


Figure 4.2: TTS(9) #30 embedded in the torus.

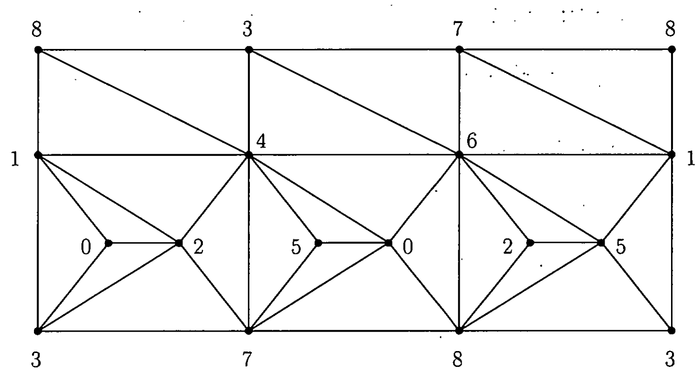


Figure 4.3: TTS(9) #31 embedded in the torus.

4.2 Maximal sets of disjoint STS(9)s

A *large set* is a collection $(V, \mathcal{B}_1), \ldots (V, \mathcal{B}_m)$ of m Steiner triple systems of order v such that every 3-subset of V is contained in at least one STS(v) of the collection. If every 3-subset of V is contained in precisely one system, i.e. $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$, then this collection is called a large set of *mutually disjoint* Steiner triple systems. An easy counting argument establishes that large sets of mutually disjoint STS(v)

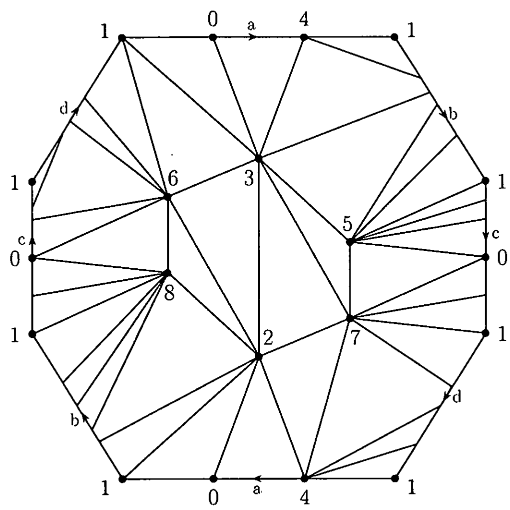


Figure 4.4: TTS(9) #34 embedded in the double torus.

contain exactly $v-2$ systems. They exist for $v \equiv 1, 3 \pmod 6), v \neq 7$ [41, 42, 43, 56]. For $v = 9$, up to isomorphism, there are two large sets of mutually disjoint Steiner triple systems [2]. These are,

	A	B	C	D	E	F	G
1.	0 1 2	0 1 3	0 1 4	0 1 5	0 1 6	0 1 7	0 1 8
	3 4 5	2 7 4	2 5 8	2 3 7	2 5 4	2 4 8	2 5 3
	6 8 7	5 6 8	6 3 7	8 6 4	7 8 3	3 6 5	4 7 6

	A	B	C	D	E	F	G
2.	0 1 2	0 1 3	0 1 4	0 1 5	0 1 6	0 1 7	0 1 8
	3 4 5	2 7 6	2 7 3	2 4 7	2 8 5	2 5 6	2 4 5
	6 8 7	8 4 5	5 8 6	6 3 8	4 3 7	3 8 4	7 6 3

Each system is represented in compact notation, e.g. the system with block set {012, 345, 678, 036, 147, 258, 048, 156, 237, 057, 138, 246} is represented as

$$\begin{matrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{matrix}$$

The STS(9) is obtained from the three horizontal triples, the three vertical triples, the three forward diagonals, and the three back diagonals. In this section we are concerned with maximal sets of mutually disjoint STS(9)s. These are sets that cannot be extended but are not necessarily large sets. There exist six nonisomorphic such sets; two of these are the large sets given above and the remaining four, which are not large sets, were found by Cooper [10], see also [33], and are given below.

3.

A	B	C	D	E
0 1 2	0 1 3	0 1 4	0 1 5	0 1 7
7 3 6	8 2 7	8 2 6	4 2 7	8 2 4
8 5 4	6 5 4	5 7 3	8 3 6	3 6 5

4.

A	B	C	D	E
0 1 2	0 1 3	0 1 4	0 1 5	0 1 6
7 3 6	8 2 7	8 2 6	3 2 7	3 2 4
8 5 4	6 5 4	5 7 3	8 4 6	5 8 7

5.

A	B	C	D	E
0 1 2	0 1 3	0 1 4	0 1 6	0 1 7
7 3 6	8 2 7	5 2 3	4 2 3	5 2 4
8 5 4	6 5 4	8 7 6	7 8 5	3 6 8

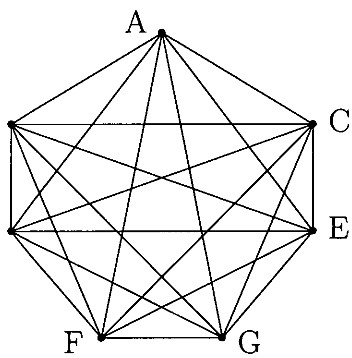
6.

A	B	C	D
0 1 2	0 1 6	0 1 7	0 1 8
7 3 6	5 2 4	4 2 3	7 2 4
8 5 4	3 7 8	8 6 5	6 5 3

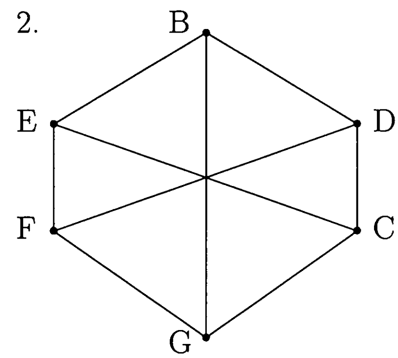
A biembedding of any pair of two disjoint STS(9)s on the same set forms a topological space which must either be an orientable pseudosurface corresponding to the embedding of the TTS(9) #33, the Emch surface, or a nonorientable surface corresponding to the embedding of the TTS(9) #36. We illustrate below how the systems of each set biembed between themselves; we do this by using graphs. Let the vertices of the graph represent the systems. Two vertices are adjacent if the corresponding systems form an Emch surface.

As can be seen, the graph of the intersections in set #1 is the complete graph K_7. Thus the intersection between every pair of STS(9)s gives the Emch surface. For set #2, the graph is the complete bipartite graph $K_{3,3}$. The seven STS(9)s can be partitioned into three sets $R = \{A\}$, $S = \{B, C, F\}$, $T = \{D, E, G\}$. The intersections between systems in set S and systems in set T give the Emch surface, but all other intersections give the nonorientable surface N_5. The graphs of set #3 and #4 do not distinguish between the two sets. Both graphs are the complete graph K_5 with a path of length 2 removed. For set #5 every pair of STS(9)s gives the Emch surface and so the graph of this set is the complete graph K_5. Finally, for set #6 again every pair gives the Emch surface with the exception of one pair.

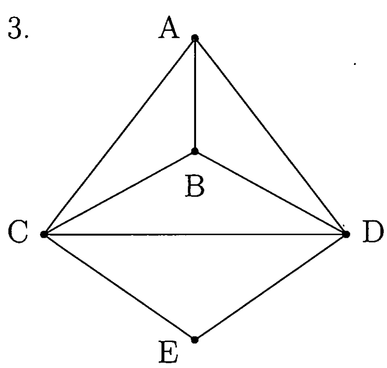
1.



2.



3.



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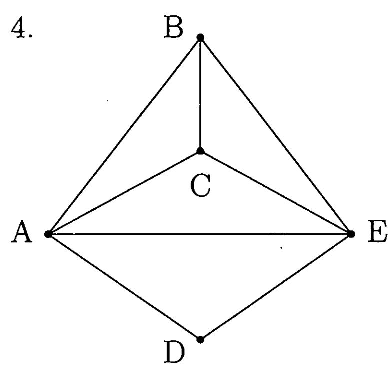

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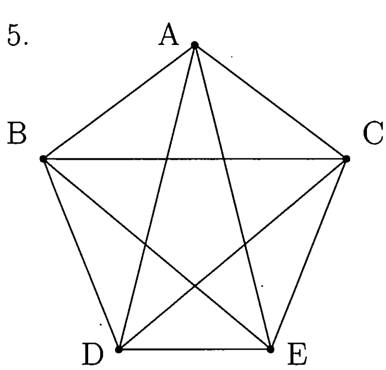
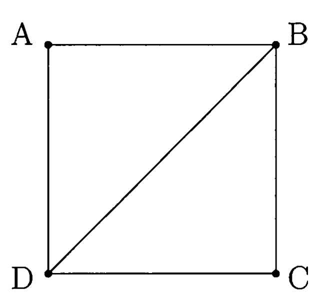


6.



CHAPTER 5

Biembeddings of idempotent Latin squares

A triangular embedding of a complete regular tripartite graph $K_{n,n,n}$ in a surface is face two-colourable if and only if the surface is orientable [23]. In this case, the faces of each colour class can be regarded as the triples of a *transversal design* $\mathrm{TD}(3,n)$, of order n and block size 3. Such a design comprises of a triple $(V,\mathcal{G},\mathcal{B})$, where V is a $3n$-element set (the points), $\mathcal{G}$ is a partition of V into three parts (the groups) each of cardinality n, and $\mathcal{B}$ is a collection of 3-element subsets (the blocks) of V such that each 2-element subset of V is either contained in exactly one block of $\mathcal{B}$, or in exactly one group of $\mathcal{G}$, but not both. Two $\mathrm{TD}(3,n)$s, $(V,\{G_1,G_2,G_3\},\mathcal{B})$ and $(V',\{G'_1,G'_2,G'_3\},\mathcal{B}')$ are said to be *isomorphic* if, for some permutation π of $\{1,2,3\}$, there exist bijections $\alpha_i : G_i \to G'_{\pi(i)}$, $i=1,2,3$, that map blocks of $\mathcal{B}$ to blocks of $\mathcal{B}'$. A Latin square of side n determines a $\mathrm{TD}(3,n)$ by assigning the row labels, the column labels, and the entries as the three groups of the design. Two Latin squares are said to be in the same *main class* if the corresponding transversal designs are isomorphic. A question that naturally arises is: which pairs of (main classes of) Latin squares may be biembedded?

This question seems to be difficult. On the existence side, recursive constructions are given in [15, 25, 29]. Of particular interest are biembeddings of Latin squares which are the Cayley tables of groups and other algebraic structures. An infinite class of biembeddings of Latin squares representing the Cayley tables of cyclic groups of order n is known for all $n \geq 2$. This is the family of regular biembeddings constructed using a voltage graph based on a dipole with n parallel edges embedded in a sphere [55], or alternatively directly from the Latin squares defined by $C_n(i,j) = i + j \pmod{n}$, and $C'_n(i,j) = i + j - 1 \pmod{n}$ [23]. A regular biembedding of a Latin square of side n has the greatest possible symmetry, with full automorphism group of order $12n^2$, the maximum possible value. Recently, two other families of biembeddings of the Latin squares representing the Cayley tables of cyclic groups, also with a high degree of symmetry, have been constructed [15, 16]. Enumeration results for biembeddings of Latin squares of side 3 to 7 are given in [23] and for groups of order 8 in [24]. In [31], it was shown that with the single exception of the group C_2^2, the Cayley table of each Abelian group appears in some biembedding.

5.1 Idempotent Latin squares

In this chapter, we consider a slightly different but related aspect of biembeddings of Latin squares. Let L be a Latin square of side n, which we will think of as a set of ordered triples (i, j, k) where entry k occurs in row i, column j of L, $k = L(i, j)$. Let L' be the *transpose* of L, i.e. $(i, j, k) \in L'$ if and only if $(j, i, k) \in L$. Clearly no biembedding of L with L' exists because triples (i, i, k) occur in both squares. However, suppose that L is *idempotent*, i.e. $(i, i, i) \in L$ for all i. Denote the set of idempotent triples by I. Then it may be possible to biembed the triples $L \setminus I$ with the triples $L' \setminus I$ and it is this question which is the focus of what follows.

So, given an idempotent Latin square L of side n, we denote the set of row labels by $R = \{0_r, 1_r, \ldots, (n-1)_r\}$, the set of column labels by $C = \{0_c, 1_c, \ldots, (n-1)_c\}$, the set of entries by $E = \{0_e, 1_e, \ldots, (n-1)_e\}$, and the set of idempotent triples by $I = \{\{i_r, i_c, i_e\} : i = 0, 1, \ldots, n-1\}$. Now consider the sets of triples $L \setminus I$ (the black triples) and $L' \setminus I$ (the white triples) and glue them together along common sides, $\{i_r, j_c\}, i \neq j, \{j_c, k_e\}, j \neq k, \{k_e, i_r\}, k \neq i$. The resulting topological space is not necessarily a surface but is certainly a pseudosurface which we will call the *transpose pseudosurface* of $L \setminus I$ and denote by $S(L \setminus I)$. Within this framework, the main interest is when $S(L \setminus I)$ is a surface, in which case we say that the idempotent Latin square L biembeds with its transpose, i.e. $(L \setminus I) \bowtie (L' \setminus I)$.

From a graph theoretic viewpoint, a biembedding of an idempotent Latin square with its transpose, as described above, gives a face two-colourable triangular embedding of a complete regular tripartite graph $K_{n,n,n}$ with the removal of a triangle factor. For the same reason that applies without the removal of a triangular factor, the surface is orientable. In such a biembedding, the number of vertices, $V = 3n$, the number of edges, $E = 3(n^2 - n)$, and the number of faces, $F = 2(n^2 - n)$. Therefore, using Euler's formula, $V + F - E = 4n - n^2$ which is even if and only if n is even. In the next section, we construct biembeddings of idempotent Latin squares with their transpose for all doubly even values of n. In Section 5.3, we consider the situation when the transpose L' is mutually orthogonal to L, i.e. the Latin square L is a self-orthogonal Latin square (SOLS). Biembeddings of a self-orthogonal Latin square L with its transpose are constructed for all $n = 2^m, m \geq 2$.

The *rotation* about a point i_r is defined to be the set of cycles

$j_c^1 k_e^1 j_c^2 k_e^2 \dots j_c^{a_1-1} k_e^{a_1-1} \mid j_c^{a_1} k_e^{a_1} \dots j_c^{a_2-1} k_e^{a_2-1} \mid \dots \mid j_c^{a_{m-1}} k_e^{a_{m-1}} \dots j_c^{a_m-1} k_e^{a_m-1}$

where $k_e^s = L(i, j^s) = L'(i, j^{s+1})$, $s \in \{1, 2, \dots, n-1\} \setminus \{a_1 - 1, a_2 - 1, \dots, a_m - 1\}$ and $k_e^{a_t-1} = L(i, j^{a_t-1}) = L'(i, j^{a_{t-1}})$, $1 \leq t \leq m$, $1 \leq m \leq n-1$ with $a_0 = 1$ and $a_m = n$. The cycles are the order of vertices adjacent to i_r as determined by the biembedding. The rotation about a point j_c or k_e is defined analogously. The two Latin squares L and L' are biembedded in a surface if and only if the rotation about each point is a single cycle.

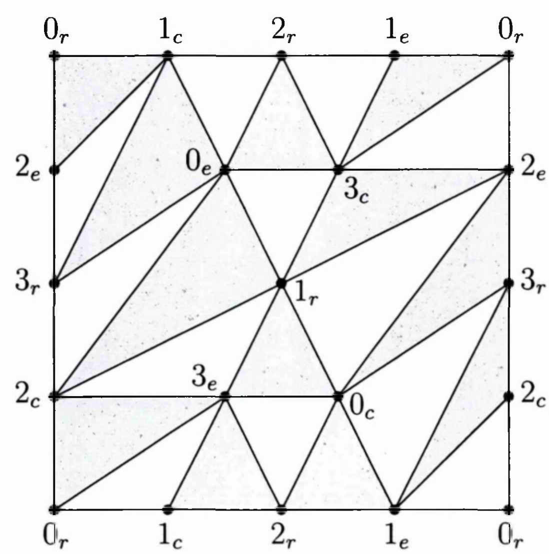
To conclude this section, below is an example which illustrates some of the ideas presented above.

Example There are two idempotent Latin squares of side 4 each of which is the transpose of the other.

	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

	0	1	2	3
0	0	3	1	2
1	2	1	3	0
2	3	0	2	1
3	1	2	0	3

These biembed in the torus as shown.



The rotation scheme is

$$\begin{array}{lll}
0_r : 1_c 2_e\, 3_c 1_e\, 2_c 3_e & 0_c : 1_e 2_r\, 3_e 1_r\, 2_e 3_r & 0_e : 1_r 2_c\, 3_r 1_c\, 2_r 3_c \\
1_r : 2_c 0_e\, 3_c 2_e\, 0_c 3_e & 1_c : 2_e 0_r\, 3_e 2_r\, 0_e 3_r & 1_e : 2_r 0_c\, 3_r 2_c\, 0_r 3_c \\
2_r : 3_c 0_e\, 1_c 3_e\, 0_c 1_e & 2_c : 3_e 0_r\, 1_e 3_r\, 0_e 1_r & 2_e : 3_r 0_c\, 1_r 3_c\, 0_r 1_c \\
3_r : 0_c 2_e\, 1_c 0_e\, 2_c 1_e & 3_c : 0_e 2_r\, 1_e 0_r\, 2_e 1_r & 3_e : 0_r 2_c\, 1_r 0_c\, 2_r 1_c
\end{array}$$

5.2 Doubly even order

In order to construct a Latin square of doubly even order which biembeds with its transpose, we use the concept of a near-Hamiltonian factorization of a complete directed graph together with known triangulations of complete (undirected) graphs in orientable surfaces. Although the main results are when the side of the Latin square $n = 4m$, $m \geq 1$, some of the theory is more general and so to begin, we do not place this restriction on n.

Let K_n (resp. K_n^*) be the complete undirected (resp. directed) graph on a set of n vertices, $\{0, 1, \ldots, n-1\}$, with $n(n-1)/2$ undirected (resp. $n(n-1)$ directed) edges. A *near-Hamiltonian circuit* of K_n^* is an ordered $(n-1)$-cycle $(x_1, x_2, \ldots, x_{n-1})$ where $x_i \neq x_j$ if $i \neq j$. A *near-Hamiltonian factorization* F of K_n^* is a partition of the directed edges of K_n^* into near-Hamiltonian circuits such that every directed edge appears in exactly one circuit. A straightforward counting argument shows that F contains n near-Hamiltonian circuits and that each vertex i, $0 \leq i \leq n-1$, is absent from precisely one circuit.

Given a near-Hamiltonian factorization F of K_n^*, an idempotent Latin square L_F of side n can be constructed as follows,

1. $L_F(i,i) = i$, $0 \leq i \leq n-1$,

2. $L_F(i,j) = k$, $0 \leq i \leq n-1$, $0 \leq j \leq n-1$, $i \neq j$, where the directed edge (i,j) occurs in the $(n-1)$-cycle which does not contain k.

Note that the above construction requires F to be a near-Hamiltonian factorization of a complete directed graph since in a near-Hamiltonian factorization of a complete undirected graph $L(i,j)$ cannot be uniquely defined. We now have the following result.

Lemma 5.2.1 *Let F be a near-Hamiltonian factorization of the complete directed graph K_n^*, and let L_F be the Latin square constructed from F as above. Let $S(L_F)$ be the transpose pseudosurface of L_F. Then the rotation about every entry point k_e, $0 \leq k \leq n-1$, is a single cycle of length $2n-2$, if n is even and two cycles each of length $n-1$ if n is odd.*

Proof Consider the near-Hamiltonian circuit not containing k. Suppose that it is $(x_1, x_2, \ldots, x_{n-1})$. Then entry k occurs in the following (row, column) pairs of L_F: $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_1), (k, k)$ and in the following (row, column) pairs of L'_F: $(x_2, x_1), (x_3, x_2), \ldots, (x_1, x_{n-1}), (k, k)$.

The rotation scheme about k_e is then

$(x_1)_r(x_2)_c(x_3)_r(x_4)_c \dots (x_{n-2})_c(x_{n-1})_r(x_1)_c(x_2)_r \dots (x_{n-2})_r(x_{n-1})_c$

if n is even, and

$(x_1)_r(x_2)_c(x_3)_r(x_4)_c \dots (x_{n-2})_r(x_{n-1})_c \mid (x_1)_c(x_2)_r(x_3)_c(x_4)_r \dots (x_{n-2})_c(x_{n-1})_r$

if n is odd. ∎

A source of near-Hamiltonian factorizations of complete directed graphs K_n^* is provided by triangulations of the complete graph K_n in an orientable surface. Triangulations in nonorientable surfaces determine near-Hamiltonian factorizations of complete undirected graphs and so are discarded. It is well-known that triangulations of K_n in orientable surfaces exist precisely when $n \equiv 0, 3, 4, 7 \pmod{12}$ [52]. Given such a triangulation on vertex set $\{0, 1, \dots, n-1\}$, first choose an arbitrary but fixed orientation. A near-Hamiltonian circuit avoiding a point is obtained by the rotation about that point in the selected orientation, and the set of all such near-Hamiltonian circuits forms a near-Hamiltonian factorization. Using this construction, we then have the following result.

Lemma 5.2.2 *Let $n \equiv 0, 3, 4, 7 \pmod{12}$, and T be a triangulation of the complete graph K_n in an orientable surface. Let $F(T)$ be the near-Hamiltonian factorization of K_n^* constructed as above. Let $L_{F(T)}$ be the Latin square constructed from $F(T)$ and $S(L_{F(T)})$ the transpose pseudosurface of $L_{F(T)}$. Then the rotation about every row point i_r, $0 \le i \le n-1$, and every column point j_c, $0 \le j \le n-1$, is a single cycle of length $2n-2$, if n is even, and two cycles each of length $n-1$ if n is odd.*

Proof The Latin square L constructed from the triangulation T has the property that if $L(i,j) = k$ then $L(j,k) = i$ and $L(k,i) = j$. It follows that the rotation about a row point i_r (resp. column point j_c) can be obtained from the rotation about i_e (resp. j_e) by applying the permutations $(e\ r\ c)$ (resp. $(e\ c\ r)$) to the suffices. ∎

The example in Section 5.1 shows the biembedding of the idempotent Latin square of order 4 with its transpose; this Latin square can be obtained by using the rotation scheme of the triangular embedding of K_4 in the sphere which is $0:1\,2\,3,\ 1:2\,0\,3,\ 2:3\,0\,1,\ 3:0\,2\,1$. An example of the odd case is given below.

Example Consider the triangulation of the complete graph K_7 on vertex set $\{0,1,2,3,4,5,6\}$ in a torus, where the triangles are given by the sets $\{i,1+i,3+i\}$ and $\{i,3+i,2+i\}$, $0 \leq i \leq 6$.

The rotation C_i about a point i is $(1+i)\ (3+i)\ (2+i)\ (6+i)\ (4+i)\ (5+i)$. The Latin square of order 7 formed from this near-Hamiltonian factorization is

	0	1	2	3	4	5	6
0	0	3	6	2	5	1	4
1	5	1	4	0	3	6	2
2	3	6	2	5	1	4	0
3	1	4	0	3	6	2	5
4	6	2	5	1	4	0	3
5	4	0	3	6	2	5	1
6	2	5	1	4	0	3	6

Then the rotations about the various points are as follows.

$i_r : (1+i)_c(3+i)_e\,(2+i)_c(6+i)_e\,(4+i)_c(5+i)_e \;\mid\; (6+i)_c(4+i)_e(5+i)_c(1+i)_e(3+i)_c(2+i)_e$

$i_c : (1+i)_e(3+i)_r\,(2+i)_e(6+i)_r\,(4+i)_e(5+i)_r \;\mid\; (6+i)_e(4+i)_r(5+i)_e(1+i)_r(3+i)_e(2+i)_r$

$i_e : (1+i)_r(3+i)_c\,(2+i)_r(6+i)_c\,(4+i)_r(5+i)_c \;\mid\; (6+i)_r(4+i)_c(5+i)_r(1+i)_c(3+i)_r(2+i)_c$

The following theorem is now an immediate consequence of Lemmas 5.2.1 and 5.2.2.

Theorem 5.2.3 *Let $n \equiv 0,4 \pmod{12}$. Then there exists an idempotent Latin square L of side n which biembeds with its transpose, i.e. $(L \setminus I) \bowtie (L' \setminus I)$.*

In the cases where $n \equiv 3,7 \pmod{12}$, the transpose pseudosurface $S(L_{F(T)})$ constructed as in Lemma 5.2.2, although not a surface, does exhibit some regularity in that every point is a pinch point and the rotation about each point consists of two cycles each of length $n-1$ as can be seen in the previous example.

Moreover, if we separate each pinch point, the pseudosurface fractures into two orientable surfaces having isomorphic triangulations. Let $C_{i,\alpha,\beta}$, $0 \le i \le n-1$, $\alpha, \beta \in \{r, c, e\}, \alpha \ne \beta$, represent the cycle $(x_{i,1})_\alpha (x_{i,2})_\beta \dots (x_{i,n-2})_\alpha (x_{i,n-1})_\beta$. Then the rotation about a point $(x_i)_e$ is $C_{i,r,c}C_{i,c,r}$, about a point $(x_i)_r$ is $C_{i,c,e}C_{i,e,c}$, and about a point $(x_i)_c$ is $C_{i,e,r}C_{i,r,e}$. Now separate each entry point $(x_i)_e$ into two points $(x_i)_e^0$ and $(x_i)_e^1$ so that the rotation about $(x_i)_e^0$ is $C_{i,r,c}$ and about $(x_i)_e^1$ is $C_{i,c,r}$. The row and column points may then also be separated and labelled appropriately so that the rotation about $(x_i)_e^0$ is $C^0_{i,r,c}$, $(x_i)_r^0$ is $C^0_{i,c,e}$, and $(x_i)_c^0$ is $C^0_{i,e,r}$ and about $(x_i)_e^1$ is $C^1_{i,c,r}$, $(x_i)_r^1$ is $C^1_{i,e,c}$, and $(x_i)_c^1$ is $C^1_{i,r,e}$.

It remains to deal with the case $n \equiv 8 \pmod{12}$. We use a triangulation of the complete graph K_{n-1} in an orientable surface to first construct a near-Hamiltonian factorization F of K^*_{n-1} and then augment this to obtain a near-Hamiltonian factorization $\bar{F}$ of K^*_n.

Construction of $\bar{F}$

Let $n \equiv 8 \pmod{12}$. Then there exists a triangulation T of the complete graph K_{n-1} in an orientable surface, having a cyclic automorphism [52, 57, 58]. Without loss of generality assume that the vertex set of K_{n-1} is $N = \{0, 1, 2, \dots, n-2\}$ and the cyclic automorphism is generated by the mapping $i \mapsto i+1 \pmod{n-1}$. Let $F(T) = \{C_0, C_1, \dots, C_{n-2}\}$ be the near-Hamiltonian factorization of K^*_{n-1} constructed from T as above, where $C_i = ((x_1+i)\ (x_2+i)\ \dots\ (x_{n-2}+i))$, $0 \le i \le n-2$, is the near-Hamiltonian circuit which avoids the vertex i. Now choose l, $1 \le l \le n-2$, relatively prime to $n-1$, (in fact we can choose $l = 1$). Then, because T has a cyclic automorphism, there exists j, $1 \le j \le n-2$, such that $x_{j+1} - x_j = l$, where if $j = n-2$, $x_{j+1} = x_1$. Introduce a new vertex ∞. Let $\bar{C}_i = ((x_1+i)\ (x_2+i)\ \dots\ (x_j+i)\ \infty\ (x_{j+1}+i)\ \dots\ (x_{n-2}+i))$ and let $\bar{C}_\infty = (0\ l\ 2l\ \dots\ (n-2)l)$, arithmetic modulo $n-1$. Further, let $\bar{F}(T) = (\cup_{i=0}^{n-2} \bar{C}_i) \cup \bar{C}_\infty$. Then $\bar{F}(T)$ is a near-Hamiltonian factorization of K^*_n on vertex set $N \cup \{\infty\}$. We can now prove the following theorem.

Theorem 5.2.4 *Let $n \equiv 8 \pmod{12}$. Then there exists an idempotent Latin square L of side n which biembeds with its transpose, i.e.* $(L \setminus I) \bowtie (L' \setminus I)$.

Proof Let $\bar{F}(T)$ be a near-Hamiltonian factorization of the complete graph K_n obtained by the triangulation T of the complete graph K_{n-1} having a cyclic automorphism, as above. Let $L_{\bar{F}(T)}$ be the Latin square constructed from $\bar{F}(T)$ and $S(L_{\bar{F}(T)})$ the transpose pseudosurface. We need to prove that the rotations about row points, column points and entry points are all single cycles.

<u>Entry points:</u>

This follows immediately from Lemma 5.2.1.

<u>Row points:</u>

Let $x_p = l$ and $x_q = n-1-l$. The rotation about the point 0_r is

$\infty_c(x_{p+1})_e \dots (x_{n-2})_e(x_1)_c \dots (x_p)_c\infty_e(x_q)_c \dots (x_{n-2})_c(x_1)_e \dots (x_{q-1})_e$.

The rotation about the point i_r, $i \neq 0$, is obtained by adding i, modulo $n-1$. The rotation about the point ∞_r is

$0_c(x_{q-1})_e(n-1-l)_c(x_{q-1}-l)_e(n-1-2l)_c(x_{q-1}-2l)_e \dots l_c(x_{q-1}+l)_e$.

<u>Column points:</u>

With the same definition of p and q as for the row points, the rotation about the point 0_c is

$\infty_e(x_q)_r \dots (x_{n-2})_r(x_1)_e \dots (x_{q-1})_e\infty_r(x_{p+1})_e \dots (x_{n-2})_e(x_1)_r \dots (x_p)_r$.

Again the rotation about the point i_c, $i \neq 0$, is obtained by adding i, modulo $n-1$. The rotation about the point ∞_c is

$(x_{p+1})_e0_r(x_{p+1}-l)_e(n-1-l)_r(x_{p+1}-2l)_e(n-1-2l)_r \dots (x_{p+1}+l)_el_r$. ■

Example Consider the triangulation of the complete graph K_7 in a torus and the rotation C_i about a point i as given in the previous example. Choose $l = 2$. Then the rotation $\bar{C}_i$ is $(1+i)\ \infty\ (3+i)\ (2+i)\ (6+i)\ (4+i)\ (5+i)$ and $\bar{C}_\infty = (0\ 2\ 4\ 6\ 1\ 3\ 5)$. The Latin square of order 8 formed from this near-Hamiltonian factorization is

	0	1	2	3	4	5	6	∞
0	0	3	∞	2	5	1	4	6
1	5	1	4	∞	3	6	2	0
2	3	6	2	5	∞	4	0	1
3	1	4	0	3	6	∞	5	2
4	6	2	5	1	4	0	∞	3
5	∞	0	3	6	2	5	1	4
6	2	∞	1	4	0	3	6	5
∞	4	5	6	0	1	2	3	∞

The rotations about the various points are as follows.

$i_e : (1+i)_r\infty_c(3+i)_r(2+i)_c(6+i)_r(4+i)_c(5+i)_r(1+i)_c\infty_r(3+i)_c(2+i)_r(6+i)_c(4+i)_r(5+i)_c$

$\infty_e : 0_r2_c4_r6_c1_r3_c5_r0_c2_r4_c6_r1_c3_r5_c$

$i_r : (1+i)_c(3+i)_e(2+i)_c\infty_e(5+i)_c(1+i)_e(3+i)_c(2+i)_e(6+i)_c(4+i)_e\infty_c(6+i)_e(4+i)_c(5+i)_e$

$\infty_r : 0_c4_e5_c2_e3_c0_e1_c5_e6_c3_e4_c1_e2_c6_e$

$i_c : (1+i)_e(3+i)_r(2+i)_e(6+i)_r(4+i)_e\infty_r(6+i)_e(4+i)_r(5+i)_e(1+i)_r(3+i)_e(2+i)_r\infty_e(5+i)_r$

$\infty_c : 0_e1_r5_e6_r3_e4_r1_e2_r6_e0_r4_e5_r2_e3_r$

5.3 Self-orthogonal Latin squares

In this section, we present a finite field construction to biembed a self-orthogonal Latin square with its transpose in an orientable surface. First recall the definitions. In a self-orthogonal Latin square, the main diagonal is a transversal and without loss of generality, by renaming the entries, it can be assumed that L is idempotent.

The construction is not new, see for example Construction 5.44 in [20], and applies for any finite field except those of order 2 or 3. We present it in this more general form but by the calculation using Euler's formula given in Section 5.1, a biembedding can exist only for even values.

Let $\omega \notin \{0, -1, 1\}$ be a generator of the cyclic multiplicative group of $\mathrm{GF}(p^m)$. Define $L(i,j) = (i + \omega j)/(1 + \omega)$ and consequently $L'(i,j) = (j + \omega i)/(1 + \omega)$. Then it is easily verified that L is a self-orthogonal Latin square of order $n = p^m$. Suppose $x = (i+\omega j)/(1+\omega)$ and $y = (j+\omega i)/(1+\omega)$. Then for every pair (x, y)

there is a unique pair (i, j). The rows, columns, and entries of this Latin square are indexed by the elements of the Galois field, which in what follows it will be convenient to represent by $0, 1 = \omega^{n-1}, \omega, \ldots, \omega^{n-2}$.

We now restrict our attention to Galois fields GF(2^m), $m \geq 2$. By considering the rotations about each of the row, column and entry points we prove the following theorem.

Theorem 5.3.1 *Let $n = 2^m, m \geq 2$. Then there exists a self-orthogonal Latin square L of side n which biembeds with its transpose, i.e.* $(L \setminus I) \bowtie (L' \setminus I)$.

Proof Let L be the self-orthogonal Latin square of order $2^m, m \geq 2$, obtained using the finite field construction given above. Let $\zeta^k = \omega^k/(1+\omega)$, $0 \leq k \leq n-2$. Note therefore that in this context ζ^k does not have its usual meaning.

(1) Row 0 of L and column 0 of L' are as follows.

	0	1	ω	ω^2	$\ldots$	ω^{n-3}	ω^{n-2}
0	0	ζ^1	ζ^2	ζ^3	$\ldots$	ζ^{n-2}	ζ^0

Row 0 of L' and column 0 of L are as follows.

	0	1	ω	ω^2	$\ldots$	ω^{n-3}	ω^{n-2}
0	0	ζ^0	ζ^1	ζ^2	$\ldots$	ζ^{n-3}	ζ^{n-2}

Clearly the rotation about the points 0_r and 0_c are single cycles.

(2) Row ω^k of L and column ω^k of L' are as follows.

	0	$\omega^0 = 1$	ω	ω^2	$\ldots$	ω^{k-1}
ω^k	ζ^k	$\zeta^k + \zeta^1$	$\zeta^k + \zeta^2$	$\zeta^k + \zeta^3$	$\ldots$	$\zeta^k + \zeta^k = 0$

	ω^k	ω^{k+1}	$\ldots$	ω^{n-3}	ω^{n-2}
ω^k	$\zeta^k + \zeta^{k+1} = \omega^k$	$\zeta^k + \zeta^{k+2}$	$\ldots$	$\zeta^k + \zeta^{n-2}$	$\zeta^k + \zeta^0$

Row ω^k of L' and column ω^k of L are as follows.

	0	$\omega^0=1$	ω	ω^2	$\dots$	ω^{k-1}
ω^k	ζ^{k+1}	$\zeta^{k+1}+\zeta^0$	$\zeta^{k+1}+\zeta^1$	$\zeta^{k+1}+\zeta^2$	$\dots$	$\zeta^{k+1}+\zeta^{k-1}$

	ω^k	ω^{k+1}	$\dots$	ω^{n-3}	ω^{n-2}
ω^k	$\zeta^{k+1}+\zeta^k=\omega^k$	$\zeta^{k+1}+\zeta^{k+1}=0$	$\dots$	$\zeta^{k+1}+\zeta^{n-3}$	$\zeta^{k+1}+\zeta^{n-2}$

For each k, $0\le k\le n-2$, define $q_0=q_0(k)$ by the equation $\zeta^k=\zeta^{k+1}+\zeta^{q_0}$, i.e. $\omega^{q_0}=\omega^k(1-\omega)$. Further, for $1\le i\le n-2$, define $q_i=q_i(k)$ by the equations $\zeta^{q_i}=\zeta^{q_0}(1+\omega+\dots+\omega^i)$, i.e. $\omega^{q_i}=\omega^k(1-\omega^{i+1})$. Note that for $0\le i\le n-2$, the values ω^{q_i} are distinct, as are the values ζ^{q_i}. Moreover $\omega^{q_{n-2}}=0$.

The rotation about a row point ω^k_r is a single cycle as follows.

$0_c\ (\zeta^{k+1}+\zeta^{q_0})_e\ \omega^{q_0}_c\ (\zeta^{k+1}+\zeta^{q_1})_e\ \omega^{q_1}_c\ (\zeta^{k+1}+\zeta^{q_2})_e\ \dots\ \omega^{q_{n-3}}_c\ (\zeta^{k+1}+\zeta^{q_{n-2}})_e$

The rotation about a column point ω^k_c is similar and is also a single cycle.

(3) Entry 0 occurs in cells $(i,-\frac{i}{\omega})$ and in cells $(-\frac{i}{\omega},i)$, $1=\omega^0\le i\le\omega^{n-2}$, of L and L' respectively. The rotation about the point 0_e is therefore,

$1_r\ (-\frac{1}{\omega})_c\ (\frac{1}{\omega^2})_r\ (-\frac{1}{\omega^3})_c\ \dots\ (\frac{1}{\omega^{n-2}})_r\ (-\frac{1}{\omega^{n-1}})_c\ (\frac{1}{\omega})_r\ (-\frac{1}{\omega^2})_c\ \dots\ (\frac{1}{\omega^{n-3}})_r\ (-\frac{1}{\omega^{n-2}})_c$

i.e. $1_r\ (\frac{1}{\omega})_c\ (\frac{1}{\omega^2})_r\ (\frac{1}{\omega^3})_c\ \dots\ (\frac{1}{\omega^{n-2}})_r\ 1_c\ (\frac{1}{\omega})_r\ (\frac{1}{\omega^2})_c\ \dots\ (\frac{1}{\omega^{n-3}})_r\ (\frac{1}{\omega^{n-2}})_c$

which is a single cycle.

(4) Entry ω^k occurs in cells $(0,\omega^k+\omega^{k-1})$ and $(\omega^i,\omega^k+\omega^{k-1}-\omega^{i-1})$ of L and in cells $(0,\omega^k+\omega^{k+1})$ and $(\omega^k+\omega^{k-1}-\omega^{i-1},\omega^i)$ of L', $0\le i\le n-2$.

The rotation about the point ω^k_e, where k is even is

$0_r\ (\omega^k+\omega^{k-1})_c\ (\omega^k-\omega^{k-2})_r\ (\omega^k+\omega^{k-3})_c\ (\omega^k-\omega^{k-4})_r\ (\omega^k+\omega^{k-5})_c\ \dots$

$(\omega^k-\omega^2)_r\ (\omega^k+\omega)_c\ (\omega^k-1)_r\ (\omega^k+\omega^{n-2})_c\ (\omega^k-\omega^{n-3})_r\ \dots$

$(\omega^k-\omega^{k+1}=\omega^k+\omega^{k+1})_r\ 0_c\ (\omega^k+\omega^{k-1}=\omega^k-\omega^{k-1})_r(\omega^k+\omega^{k-2})_c\ \dots$

$(\omega^k-\omega)_r\ (\omega^k+1)_c\ (\omega^k-\omega^{n-2})_r\ (\omega^k+\omega^{n-3})_c\ \dots\ (\omega^k+\omega^{k+1})_c$

and where k is odd is

$0_r\ (\omega^k+\omega^{k-1})_c\ (\omega^k-\omega^{k-2})_r\ (\omega^k+\omega^{k-3})_c\ (\omega^k-\omega^{k-4})_r\ (\omega^k+\omega^{k-5})_c\ \dots$

$(\omega^k - \omega)_r\ (\omega^k + 1)_c\ (\omega^k - \omega^{n-2})_r\ (\omega^k + \omega^{n-3})_c\ (\omega^k - \omega^{n-4})_r\ \dots$

$(\omega^k - \omega^{k+1} = \omega^k + \omega^{k+1})_r\ 0_c\ (\omega^k + \omega^{k-1} = \omega^k - \omega^{k-1})_r (\omega^k + \omega^{k-2})_c\ \dots$

$(\omega^k - \omega^2)_r\ (\omega^k + \omega)_c\ (\omega^k - 1)_r\ (\omega^k + \omega^{n-2})_c\ \dots\ (\omega^k + \omega^{k-1})_c$

in either case, a single cycle. ■

It is worth remarking that for a Galois field GF(p^m) where p is an odd prime and $m \geq 1$, except for $(p, m) = (3, 1)$, the rotation about all row points and all column points is also a single cycle. The proof is precisely as given above for GF(2^m), $m \geq 2$, except for the observations that $\zeta^k + \zeta^k = \zeta^{k+1} + \zeta^{k+1} = 0$ which in fact play no part in the proof.

However the proof that the rotation about all entry points is a single cycle, does rely on the field having characteristic 2. Otherwise, we find that the rotation about all entry points is two cycles each of length $p^m - 1$. Thus in these cases, although the transpose pseudosurface $S(L \setminus I)$ is not a surface, it does exhibit some regularity.

The first example of this chapter gives a self-orthogonal Latin square which can be biembedded with its transpose. A further example is given below.

Example Let $F = \mathrm{GF}(2^3)$ with irreducible polynomial $x^3 = x + 1$ and choose $\omega = x$. Then $(x, x^2, x^3, x^4, x^5, x^6, x^7) = (x, x^2, x+1, x^2+x, x^2+x+1, x^2+1, 1)$. Then the Latin square L, obtained from the construction described in this section is,

	0	1	x	x^2	$x+1$	x^2+x	x^2+x+1	x^2+1
0	0	x^2+x+1	x^2+1	1	x	x^2	$x+1$	x^2+x
1	x^2+x	1	$x+1$	x^2+x+1	x^2	x	x^2+1	0
x	x^2+x+1	0	x	x^2+x	x^2+1	$x+1$	x^2	1
x^2	x^2+1	x	0	x^2	x^2+x+1	1	x^2+x	$x+1$
$x+1$	1	x^2+x	x^2	0	$x+1$	x^2+1	x	x^2+x+1
x^2+x	x	x^2+1	x^2+x+1	$x+1$	0	x^2+x	1	x^2
x^2+x+1	x^2	$x+1$	1	x^2+1	x^2+x	0	x^2+x+1	x
x^2+1	$x+1$	x^2	x^2+x	x	1	x^2+x+1	0	x^2+1

The rotation scheme is

$$0_r : 1_c(x^2+x+1)_e x_c(x^2+1)_e x_c^2 1_e(x+1)_c x_e(x^2+x)_c x_e^2(x^2+x+1)_c(x+1)_e(x^2+1)_c(x^2+x)_e$$

$$1_r : 0_c(x^2+x)_e(x+1)_c x_e^2(x^2+1)_c 0_e x_c(x+1)_e(x^2+x+1)_c(x^2+1)_e(x^2+x)_c x_e x_c^2(x^2+x+1)_e$$

$$x_r : (x+1)_c(x^2+1)_e 0_c(x^2+x+1)_e(x^2+x)_c(x+1)_e 1_c 0_e x_c^2(x^2+x)_e(x^2+1)_c 1_e(x^2+x+1)_c x_e^2$$

$$x_r^2 : (x^2+1)_c(x+1)_e(x^2+x)_c 1_e 0_c(x^2+1)_e(x^2+x+1)_c(x^2+x)_e x_c 0_e(x+1)_c(x^2+x+1)_e 1_c x_e$$

$$(x+1)_r : x_c x_e^2 1_c(x^2+x)_e(x^2+x+1)_c x_e 0_c 1_e(x^2+1)_c(x^2+x+1)_e x_c^2 0_e(x^2+x)_c(x^2+1)_e$$

$$(x^2+x)_r : (x^2+x+1)_c 1_e x_c^2(x+1)_e x_c(x^2+x+1)_e(x^2+1)_c x_e^2 0_c x_e 1_c(x^2+1)_e(x+1)_c 0_e$$

$$(x^2+x+1)_r : (x^2+x)_c 0_e(x^2+1)_c x_e(x+1)_c(x^2+x)_e x_c^2(x^2+1)_e 1_c(x+1)_e 0_c x_e^2 x_c 1_e$$

$$(x^2+1)_r : x_c^2 x_e(x^2+x+1)_c 0_e 1_c x_e^2(x^2+x)_c(x^2+x+1)_e(x+1)_c 1_e x_c(x^2+x)_e 0_c(x+1)_e$$

$$0_c : 1_r(x^2+x)_e(x^2+1)_r(x+1)_e(x^2+x+1)_r x_e^2(x^2+x)_r x_e(x+1)_r 1_e x_r^2(x^2+1)_e x_r(x^2+x+1)_e$$

$$1_c : 0_r(x^2+x+1)_e x_r^2 x_e(x^2+x)_r(x^2+1)_e(x^2+x+1)_r(x+1)_e x_r 0_e(x^2+1)_r x_e^2(x+1)_r(x^2+x)_e$$

$$x_c : (x+1)_r x_e^2(x^2+x+1)_r 1_e(x^2+1)_r(x^2+x)_e x_r^2(x+1)_e 1_r(x+1)_e(x^2+x)_r(x^2+x+1)_e 0_r(x^2+1)_e$$

$$x_c^2 : (x^2+1)_r x_e 1_r(x^2+x+1)_e(x+1)_r 0_e x_r(x^2+x)_e(x^2+x+1)_r(x^2+1)_e 0_r 1_e(x^2+x)_r(x+1)_e$$

$$(x+1)_c : x_r(x^2+1)_e(x^2+x)_r 0_e x_r^2(x^2+x+1)_e(x^2+1)_r 1_e 0_r x_e(x^2+x+1)_r(x^2+x)_e 1_r x_e^2$$

$$(x^2+x)_c : (x^2+x+1)_r 0_e(x+1)_r(x^2+1)_e 1_r x_e 0_r x_e^2(x^2+1)_r(x^2+x+1)_e x_r(x+1)_e x_r^2 1_e$$

$$(x^2+x+1)_c : (x^2+x)_r 1_e x_r x_e^2 0_r(x+1)_e 1_r(x^2+1)_e x_r^2(x^2+x)_e(x+1)_r x_e(x^2+1)_r 0_e$$

$$(x^2+1)_c : (x_r^2(x+1)_e 0_r(x^2+x)_e x_r 1_e(x+1)_r(x^2+x+1)_e(x^2+x)_r x_e^2 1_r 0_e(x^2+x+1)_r x_e$$

$$0_e : 1_r(x^2+1)_c(x^2+x+1)_r(x^2+x)_c(x+1)_r x_c^2 x_r 1_c(x^2+1)_r(x^2+x+1)_c(x^2+x)_r(x+1)_c x_r^2 x_c$$

$$1_e : 0_r x_c^2(x^2+x)_r(x^2+x+1)_c x_r(x^2+1)_c(x+1)_r 0_c x_r^2(x^2+x)_c(x^2+x+1)_r x_c(x^2+1)_r(x+1)_c$$

$$x_e : (x+1)_r(x^2+x+1)_c(x^2+1)_r x_c^2 1_r(x^2+x)_c 0_r(x+1)_c(x^2+x+1)_r(x^2+1)_c x_r^2 1_c(x^2+x)_r 0_c$$

$$x_e^2 : (x^2+1)_r 1_c(x+1)_r x_c(x^2+x+1)_r 0_c(x^2+x)_r(x^2+1)_c 1_r(x+1)_c x_r(x^2+x+1)_c 0_r(x^2+x)_c$$

$$(x+1)_e : x_r(x^2+x)_c x_r^2(x^2+1)_c 0_r(x^2+x+1)_c 1_r x_c(x^2+x)_r x_c^2(x^2+1)_r 0_c(x^2+x+1)_r 1_c$$

$$(x^2+x)_e : (x^2+x+1)_r(x+1)_c 1_r 0_c(x^2+1)_r x_c x_r^2(x^2+x+1)_c(x+1)_r 1_c 0_r(x^2+1)_c x_r x_c^2$$

$$(x^2+x+1)_e : (x^2+x)_r x_c 0_r 1_c x_r^2(x+1)_c(x^2+1)_r(x^2+x)_c x_r 0_c 1_r x_c^2(x+1)_r(x^2+1)_c$$

$$(x^2+1)_e : x_r^2 0_c x_r(x+1)_c(x^2+x)_r 1_c(x^2+x+1)_r x_c^2 0_r x_c(x+1)_r(x^2+x)_c 1_r(x^2+x+1)_c$$

CHAPTER 6

Maximum genus embeddings of Latin squares

In this chapter we investigate a different problem concerning embeddings of Latin squares. We first recall some definitions from the previous chapter. A triangular embedding of a complete tripartite graph $K_{n,n,n}$ is face two-colourable if and only if the supporting surface is orientable. In such a case, the faces of each colour class can be regarded as the triples of a *transversal design* $\mathrm{TD}(3, n)$, of order n and block size 3. A Latin square of side n determines a $\mathrm{TD}(3, n)$ by assigning the rows, the columns, and the entries as the three groups of the design. The two Latin squares are said to be *biembedded* in the surface. Whenever such a biembedding exists, it represents a face two-colourable embedding of $K_{n,n,n}$ in a surface of minimum genus.

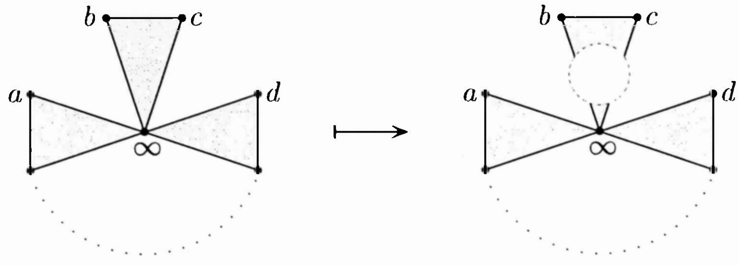
The purpose of this chapter is to investigate the opposite case, namely the embeddings of Latin squares in surfaces of maximum genus. To be precise, we seek a face two-colourable embedding of $K_{n,n,n}$ in a surface in which the faces in one of the two colour classes are triangles and so determine a Latin square of order n, while there is just one face in the second colour class and the interior of that face is homeomorphic to an open disc. We call this an *upper embedding* of the Latin square. These types of embeddings have already been investigated for Steiner triple systems in [28].

6.1 Existence of upper embeddings

Consider a Latin square of order n. In an upper embedding the number of vertices (V) is $3n$, the number of edges (E) is $3n^2$ and the number of faces (F) is $n^2 + 1$. So $V + F - E = 1 + 3n - 2n^2$. For a nonorientable upper embedding the genus $\gamma = (2n - 1)(n - 1)$ whilst for an orientable upper embedding the genus $g = (2n - 1)(n - 1)/2$ which requires that in this case n must be odd. We first consider the nonorientable case and prove the following theorem.

Theorem 6.1.1 *Every Latin square has an upper embedding in a nonorientable surface.*

Proof Begin with any face two-colourable embedding of $K_{n,n,n}$ in which the black faces are triangles representing the Latin square. If there is just one white face then we have an upper embedding. Otherwise, there exists at least one black triangle that is incident to two white faces. With the addition of a crosscap across the black triangle we join these two white faces together, reducing the number of faces by one and increasing the nonorientable genus by one; this is shown in the figure below. By repetition of this procedure we obtain a nonorientable upper embedding of the Latin square. ■



$\infty : \ldots a\underline{bc}d \ldots$ $\qquad$ $\infty : \ldots a\underline{cb}d \ldots$

Figure 6.1: Joining two white faces.

The process is illustrated in the following example of a nonorientable embedding of the Latin square which is the Cayley table of the Klein group K_4.

Example

	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Consider the embedding of the above Latin square with the following rotation scheme.

$$\begin{array}{lll} 0_r : 0_c0_e\ 3_c3_e\ 2_c2_e\ 1_c1_e & 0_c : 0_e0_r\ 3_e3_r\ 2_e2_r\ 1_e1_r & 0_e : 0_r0_c\ 3_r3_c\ 1_r1_c\ 2_r2_c \\ 1_r : 0_c1_e\ 3_c2_e\ 2_c3_e\ 1_c0_e & 1_c : 1_e0_r\ 2_e3_r\ 3_e2_r\ 0_e1_r & 1_e : 1_r0_c\ 2_r3_c\ 3_r2_c\ 0_r1_c \\ 2_r : 0_c2_e\ 3_c1_e\ 2_c0_e\ 1_c3_e & 2_c : 2_e0_r\ 1_e3_r\ 0_e2_r\ 3_e1_r & 2_e : 2_r0_c\ 1_r3_c\ 3_r1_c\ 0_r2_c \\ 3_r : 0_c3_e\ 3_c0_e\ 2_c1_e\ 1_c2_e & 3_c : 3_e0_r\ 1_e2_r\ 0_e3_r\ 2_e1_r & 3_e : 3_r0_c\ 0_r3_c\ 2_r1_c\ 1_r2_c \end{array}$$

This embedding has 16 black triangular faces and 4 white faces. The white faces are:

W_1: $0_e0_r3_c1_e3_r1_c3_e1_r1_c1_e1_r3_c3_e2_r0_c1_e2_r2_c3_e3_r3_c2_e3_r0_c2_e1_r2_c2_e2_r3_c0_e1_r0_c0_e3_r2_c$

W_2: $0_r0_c3_e0_r2_c1_e$

W_3: $0_r1_c2_e$

W_4: $0_e2_r1_c$

Adding a crosscap across the black triangle $\{2_r, 2_c, 0_e\}$ joins the two white faces W_1 and W_4. Similarly, adding a crosscap across the black triangle $\{0_r, 2_c, 2_e\}$ joins the white faces W_2 and W_3. The two new white faces are:

W_A: $0_e0_r3_c1_e3_r1_c3_e1_r1_c1_e1_r3_c3_e2_r0_c1_e2_r0_e1_c2_r2_c3_e3_r3_c2_e3_r0_c2_e1_r2_c2_e2_r3_c0_e1_r0_c0_e3_r2_c$

W_B: $0_r2_e1_c0_r2_c1_e0_r0_c3_e$

Finally, the addition of a crosscap across the black triangle $\{3_r, 2_c, 1_e\}$ joins the two remaining white faces W_A and W_B together and the resulting embedding is nonorientable and of maximum genus with rotation scheme,

$$\begin{array}{lll} 0_r : 0_c0_e\ 3_c3_e\ 2_e2_c\ 1_c1_e & 0_c : 0_e0_r\ 3_e3_r\ 2_e2_r\ 1_e1_r & 0_e : 0_r0_c\ 3_r3_c\ 1_r1_c\ 2_r2_c \\ 1_r : 0_c1_e\ 3_c2_e\ 2_c3_e\ 1_c0_e & 1_c : 1_e0_r\ 2_e3_r\ 3_e2_r\ 0_e1_r & 1_e : 1_r0_c\ 2_r3_c\ 2_c3_r\ 0_r1_c \\ 2_r : 0_c2_e\ 3_c1_e\ 0_e2_c\ 1_c3_e & 2_c : 2_e0_r\ 1_e3_r\ 0_e2_r\ 3_e1_r & 2_e : 2_r0_c\ 1_r3_c\ 3_r1_c\ 0_r2_c \\ 3_r : 0_c3_e\ 3_c0_e\ 2_c1_e\ 1_c2_e & 3_c : 3_e0_r\ 1_e2_r\ 0_e3_r\ 2_e1_r & 3_e : 3_r0_c\ 0_r3_c\ 2_r1_c\ 1_r2_c \end{array}$$

and large face,

$0_e\ 0_r\ 3_c\ 1_e\ 2_c\ 0_r\ 1_c\ 2_e\ 0_r\ 3_e\ 0_c\ 0_r\ 1_e\ 3_r\ 1_c\ 3_e\ 1_r\ 1_c\ 1_e\ 1_r\ 3_c\ 3_e\ 2_r\ 0_c\ 1_e\ 2_r\ 0_e\ 1_c\ 2_r\ 2_c$
$3_e\ 3_r\ 3_c\ 2_e\ 3_r\ 0_c\ 2_e\ 1_r\ 2_c\ 2_e\ 2_r\ 3_c\ 0_e\ 1_r\ 0_c\ 0_e\ 3_r\ 2_c$

We now turn our attention to orientable surfaces. As we showed in the beginning of the chapter, orientable embeddings require the order of the Latin square to be odd. Despite this arithmetic restriction, the proof for the existence of orientable upper embeddings is much more involved compared to the nonorientable case.

Theorem 6.1.2 *Every Latin square of odd order n has an upper embedding in an orientable surface.*

Proof The triples $\{i_r, j_c, k_e\}$, $i, j, k \in \mathbb{Z}_n$, of the Latin square will be represented as the black triangles of the embedding. Choose a fixed row point x_r and a fixed column point y_c. Take the triangle T containing both of these points together with a further $(n-1)/2$ triangles containing x_r and a further $(n-1)/2$ triangles containing y_c such that, together with T, these n triangles contain all n entry points. Represent these triangles on a sphere. Note that the boundary of the large face contains all entry points. Now take the $(n-1)/2$ row points which are not contained in any taken triangle and the $(n-1)/2$ untaken column points and pair them arbitrarily. Attach the triangles containing these pairs to the spherical embedding at the appropriate entry points. This procedure gives a spherical embedding containing $(3n-1)/2$ black triangles and one white face with every row, column and entry point occurring at least once on its boundary. Note also that the black triangles can be oriented in such a way that the points on the boundary follow the sequence $i_r\ j_c\ k_e\ \ldots$

We now proceed to add the remaining $(2n^2-3n+1)/2$ triples of the Latin square, one at a time, increasing the genus by one at each step. Consider at any stage the boundary of the white face. We will use the fact that every point of the Latin square appears on the boundary at least once. This assumption is certainly true for the initial embedding described above. If the next triple to be added is $\{u_r, v_c, w_e\}$ then we locate one occurrence of each of these points on the boundary of the white face, add a handle to the white face, and paste on the triangle (u_r, v_c, w_e).

If the points u_r, v_c, w_e originally divided the boundary of the white face into three sections A, B and C, e.g. $Av_cBw_eCu_r$, then it is easy to see that, after the addition of the black triangle (u_r, v_c, w_e) there still remains only one white face with boundary A (v_cw_e) C (u_rv_c) B (w_eu_r) (see Figure 6.2). This face has three more edges than at the previous stage and every point of the Latin square still appears on the boundary. It is also clear that if the interior of the white face was homeomorphic to an open disc prior to the addition of the black triangle, then it would remain so after this addition. ■

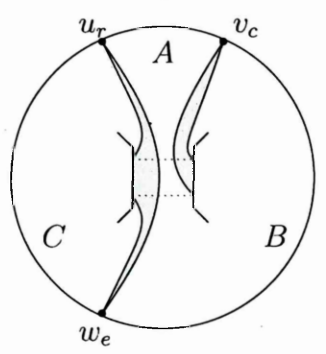
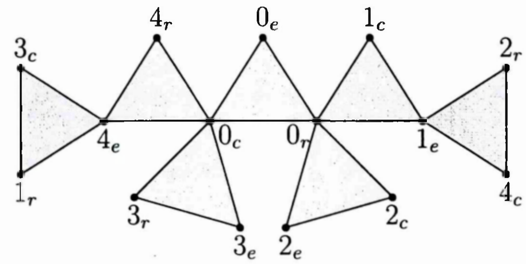


Figure 6.2: Adding a black triangle.

Example Consider the main class of the non-cyclic Latin square of order 5.

	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	3	4	0	1
3	3	4	1	2	0
4	4	2	0	1	3

First take the triangles $\{0_r, 0_c, 0_e\}$, $\{0_r, 1_c, 1_e\}$, $\{0_r, 2_c, 2_e\}$, $\{3_r, 0_c, 3_e\}$, $\{4_r, 0_c, 4_e\}$, $\{1_r, 3_c, 4_e\}$, $\{2_r, 4_c, 1_e\}$ and represent them in a sphere as shown below.



Then the large face of the initial embedding is:

$0_r\ \underline{1_c\ 1_e\ 2_r\ 4_c\ 1_e\ 0_r\ 2_c\ 2_e\ 0_r\ 0_c}\ 3_e\ \underline{3_r\ 0_c\ 4_e\ 1_r}\ 3_c\ \underline{4_e\ 4_r\ 0_c\ 0_e}$

Finally, add the remaining 18 triples of the Latin square, one at a time. At each stage, we give the boundary of the large face. The underlined sections are the sections that divide the boundary in order to accommodate the addition of the new triangle.

$\{0_r, 3_c, 3_e\}$ added; the large face is:

$0_r\ \underline{0_e\ 0_c\ 4_r\ 4_e\ 3_c\ 3_e\ 0_c\ 0_r\ 2_e\ 2_c\ 0_r\ 1_e}\ 4_c\ \underline{2_r\ 1_e\ 1_c\ 0_r\ 3_c\ 1_r}\ 4_e\ \underline{0_c\ 3_r\ 3_e}$

$\{0_r, 4_c, 4_e\}$ added; the large face is:

$\underline{0_r\ 0_e}\ 0_c\ \underline{4_r\ 4_e\ 3_c\ 3_e\ 0_c\ 0_r\ 2_e\ 2_c\ 0_r}\ 1_e\ \underline{4_c\ 4_e\ 0_c\ 3_r\ 3_e\ 0_r\ 4_c\ 2_r\ 1_e\ 1_c\ 0_r\ 3_c}\ 1_r\ \underline{4_e}$

$\{1_r, 0_c, 1_e\}$ added; the large face is:

$1_r\ \underline{4_e\ 0_r}\ 0_e\ \underline{0_c\ 1_e\ 4_c\ 4_e\ 0_c\ 3_r\ 3_e\ 0_r\ 4_c\ 2_r\ 1_e}\ 1_c\ \underline{0_r\ 3_c\ 1_r\ 0_c\ 4_r\ 4_e\ 3_c\ 3_e\ 0_c\ 0_r\ 2_e\ 2_c\ 0_r\ 1_e}$

$\{1_r, 1_c, 0_e\}$ added; the large face is:

$1_r\ \underline{1_e\ 0_r}\ 2_c\ \underline{2_e\ 0_r\ 0_c}\ 3_e\ \underline{3_c\ 4_e\ 4_r\ 0_c\ 1_r\ 3_c\ 0_r\ 1_c\ 0_e\ 0_r\ 4_e\ 1_r\ 1_c\ 1_e\ 2_r\ 4_c\ 0_r\ 3_e\ 3_r\ 0_c\ 4_e}$
$\underline{4_c\ 1_e}\ \underline{0_c\ 0_e}$

$\{1_r, 2_c, 3_e\}$ added; the large face is:

$1_r\ \underline{1_e\ 0_r\ 2_c\ 3_e\ 3_c\ 4_e\ 4_r\ 0_c\ 1_r\ 3_c\ 0_r\ 1_c\ 0_e\ 0_r\ 4_e\ 1_r\ 1_c\ 1_e\ 2_r}\ 4_c\ \underline{0_r\ 3_e\ 3_r\ 0_c\ 4_e\ 4_c\ 1_e\ 0_c\ 0_e}$
$\underline{1_r}\ \underline{2_c}\ 2_e\ \underline{0_r\ 0_c\ 3_e}$

$\{1_r, 4_c, 2_e\}$ added; the large face is,

$\underline{1_r\ 1_e\ 0_r\ 2_c\ 3_e\ 3_c\ 4_e\ 4_r\ 0_c\ 1_r\ 3_c\ 0_r\ 1_c\ 0_e\ 0_r\ 4_e\ 1_r\ 1_c\ 1_e}\ 2_r\ \underline{4_c\ 2_e\ 0_r}\ 0_c\ \underline{3_e\ 1_r\ 4_c\ 0_r\ 3_e}$
$\underline{3_r\ 0_c}\ \underline{4_e\ 4_c\ 1_e\ 0_c\ 0_e\ 1_r\ 2_c}\ 2_e$

By repetition of this technique, add the remaining triangles in this order:

$\{2_r, 0_c, 2_e\}$, $\{2_r, 1_c, 3_e\}$, $\{2_r, 2_c, 4_e\}$, $\{2_r, 3_c, 0_e\}$, $\{3_r, 1_c, 4_e\}$, $\{3_r, 2_c, 1_e\}$, $\{3_r, 3_c, 2_e\}$, $\{3_r, 4_c, 0_e\}$, $\{4_r, 1_c, 2_e\}$, $\{4_r, 2_c, 0_e\}$, $\{4_r, 3_c, 1_e\}$ and $\{4_r, 4_c, 3_e\}$.

Then the large face of the upper embedding is,

$4_r\ 4_e\ 2_c\ 0_e\ 1_c\ 2_e\ 3_c\ 1_e\ 2_c\ 0_r\ 1_e\ 1_r\ 2_e\ 0_c\ 0_r\ 2_e\ 4_c\ 3_e\ 0_r\ 4_c\ 1_r\ 3_e\ 1_c\ 0_r\ 3_c\ 2_r\ 4_e\ 3_c\ 3_r\ 1_e$
$1_c\ 1_r\ 4_e\ 1_c\ 2_r\ 2_e\ 4_r\ 2_c\ 3_r\ 4_e\ 0_r\ 0_e\ 4_r\ 3_c\ 3_e\ 2_c\ 2_r\ 3_e\ 0_c\ 2_r\ 1_e\ 4_r\ 4_c\ 2_r\ 0_e\ 4_c\ 4_e\ 0_c\ 3_r$
$0_e\ 0_c\ 1_e\ 4_c\ 3_r\ 2_e\ 2_c\ 1_r\ 0_e\ 3_c\ 1_r\ 0_c\ 4_r\ 1_c\ 3_r\ 3_e$

6.2 Automorphisms

Throughout the rest of this chapter we investigate possible automorphisms of an orientable upper embedding of a Latin square. Each of the three sets of row points, column points and entry points will be called a *part*. We first prove a fairly easy theorem.

Theorem 6.2.1 *Let ϕ be an automorphism of an orientable upper embedding of a Latin square of order n. If ϕ is not the identity automorphism then it has fixed points from only one part.*

Proof Suppose that ϕ has two fixed points, a and b, each from different parts. Then there is an edge ab which is fixed by ϕ. Therefore, by considering the large face, ϕ fixes every point and is the identity. ■

Automorphisms may be either orientation-preserving or orientation-reversing. We will first show that orientation-reversing automorphisms do not exist. Any such automorphism will act on the boundary of the large face as a reflection across an axis. Since there is an odd number of points on the boundary, this axis will pass through exactly one point, say 0_r, and exactly one edge; thus the automorphism will be an involution having a single fixed point 0_r. Now consider the triangles containing the point 0_r. There is an odd number of these and so one of them, without loss of generality $\{0_r, 0_c, 0_e\}$, must be fixed. Hence, the transposition $(0_c\ 0_e)$ is part of the involution. Consequently, the automorphism will map a row point to a row point, a column point to an entry point and an entry point to a column point. Therefore, such an automorphism will be of the form $(0_r)\,((x_1)_r\ (x_2)_r)\ \ldots\ ((x_{n-2})_r\ (x_{n-1})_r)\,(0_c\ 0_e)\,((y_1)_c\ (z_1)_e)\ \ldots\ ((y_{n-1})_c\ (z_{n-1})_e)$ where $x_i, y_i, z_i \in \mathbb{Z}_n^*, x_i \neq x_j$ if $i \neq j$. It further follows that the edge through which the axis passes is of the form $\{\alpha_c, \beta_e\}$. However, we show that such an automorphism cannot exist.

Theorem 6.2.2 *Orientation-reversing automorphisms of an orientable upper embedding of $K_{n,n,n}$ do not exist.*

Proof Assume that such an automorphism does exist and that it is of the form given above. Now assume that this automorphism maps u_c to v_e and vice versa, where $u \neq \alpha$ and $v \neq \beta$. Then the edge $\{u_c, v_e\}$ exists somewhere on the boundary of the large face, say on the right side of the axis. Since this automorphism is a reflection, the edge $\{v_e, u_c\}$ must exist on the left side of the axis. This means the same edge appears twice on the boundary of the large face, a contradiction. ∎

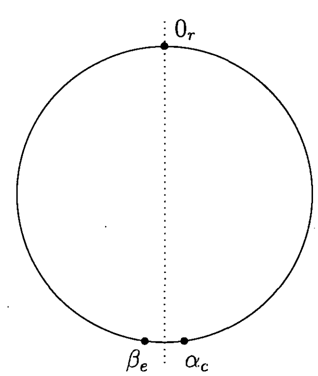


Figure 6.3: Orientation-reversing automorphism.

So all automorphisms are orientation-preserving and we now consider these. Since the action of any such automorphism on the boundary of the large face is a rotation, the group is cyclic and its order must divide $3n^2$, the number of edges in the large face. Orientation-preserving automorphisms will be of three types:

1. those that preserve all three parts,

2. those that fix one part and interchange the other two,

3. those that cyclically permute all three parts.

Consider first orientation-preserving automorphisms that preserve all three parts (the other two types will be dealt with later). Let G be the group of these automorphisms. Then as observed above $G = \mathbb{Z}_m$ and $m \mid 3n^2$. However, since G preserves all three parts $m \mid n^2$. But in fact we can prove that $m \mid n$.

Theorem 6.2.3 *Let $G = \mathbb{Z}_m$ be the group of orientation and part preserving automorphisms of the orientable upper embedding. Then $m \mid n$.*

Proof Let $n > 1$ and denote the orientable upper embedding of $K_{n,n,n}$ by M. To obtain further restrictions on m we will replace M with a related map on which G will act freely and use elementary theory of regular coverings.

Let T be the truncation of M. *Truncation* refers to the substitution of every vertex v of the embedding by a cycle of order $deg(v)$. This truncation has $3n$ *yellow* faces of length $2n$ that arise by truncating each of the $3n$ vertices of the original map, n^2 *green* faces of length 6 that arise from the n^2 triangular faces of M, and one *white* face of length $6n^2$ arising from the large face of M. The cyclic group G clearly acts freely on the vertex set of T. Since G preserves each part of $K_{n,n,n}$ and no triangle of $K_{n,n,n}$ has two vertices from the same part, no two distinct vertices of a yellow face in T can be mapped onto each other by the action of G on T.

Consider now the quotient map $M' = T/G$ whose vertices, edges and faces are G-orbits of the vertices, edges and faces of T. The conclusion of the previous paragraph implies that M' has n^2/m *green* hexagonal faces arising from the n^2 green faces of T. The action of G on the white face of T leaves one *white* face of length $6n^2/m$ in M'. To determine what happens to the $3n$ yellow faces of length $2n$ in T when passing to the quotient M', for each vertex v of $K_{n,n,n}$ let G_v be the stabilizer of v in the action of G on vertices of $K_{n,n,n}$ and let $|G_v| = m_v$. Being a subgroup of a cyclic group, each G_v must be cyclic, and the natural covering $T \to M'$ maps a G-orbit consisting of m/m_v yellow faces in T onto a single *yellow* face in M' of length $2n/m_v$. As an aside, observe that m_v must be a divisor of n, since G_v acts freely as a cyclic group of order m_v on the n triangular faces of the original map M incident with v.

By Theorem 2.2.2 of [34] we know that the regular covering $T \to M'$ induced by the free action of G can be reconstructed by means of a lift with the help of an ordinary voltage assignment α on the darts of M' in the group G. In the reconstruction process we will use elementary properties of regular coverings as listed in [34]. The net voltage on each of the n^2/m green faces of M' must be zero, as each of them lifts onto m green faces of T of the same length. For each vertex v of $K_{n,n,n}$, the yellow face of M' of length $2n/m_v$ lifts onto m/m_v yellow faces of T of length $2n$. Therefore, the net voltage on such a yellow face of M' must be an element of G of order m_v. Finally, the net voltage on the white face of M' must be an element of G of order m because this face of length $6n^2/m$ lifts onto the white face of T of length $6n^2$. Here and in what follows we assume that all the net voltages are calculated with respect to a fixed orientation of the supporting surface of the map M'. Of course, the net voltages in our case do not depend on choosing the initial point on a cycle because our voltage group G is Abelian.

Since the sum of the net voltages on all faces of M' is zero, the above analysis implies the negative of the net voltage w on the white face is equal to the sum S of the net voltages on all yellow faces of M'. The element w has order m in G, hence so has S. Observe that all summands in S have orders m_v where v ranges over a set O of representatives of the orbits of G on the vertex set of $K_{n,n,n}$. But the elements of $G \cong \mathbb{Z}_m$ of order m_v have precisely the form $(m/m_v)t_v$ where $\gcd(m_v, t_v) = 1$ and $1 \le t_v < m_v$.

It follows that S can be expressed in the form

$$S = \sum_{v \in O} \frac{m}{m_v} t_v$$

for some t_v as above, where $v \in O$. Let $m = ac$ and $n = bc$ with $\gcd(a, b) = 1$.

Recalling that $m_v \mid n$ for each $v \in O$, we have

$$S = \sum_{v \in O} \frac{m}{m_v} t_v = \frac{m}{n} \sum_{v \in O} \frac{n}{m_v} t_v = \frac{a}{b} j$$

where j is an integer mod m. It follows that S is an a-multiple of an element of $\mathbb{Z}_m$. The order of S however ought to be m, and as $a \mid m$, this is possible only if $a = 1$. Consequently, $m \mid n$, as claimed. ■

We now need the following definition. A group G of permutations of a set S is semi-regular if for any two elements $x, y \in S$, there exists at most one element $g \in G$ such that $y = gx$.

Lemma 6.2.4 *The cyclic group $\mathbb{Z}_m$, where $m \mid n$, is not semi-regular on all three parts of the embedding.*

Proof Suppose that $\mathbb{Z}_m$ is semi-regular on each part of the embedding. The quotient of the embedding under the action of $\mathbb{Z}_m$ is $K^m_{n/m,n/m,n/m}$, i.e. embedded with n^2/m triangles and one large face of length $3n^2/m$. It follows from Theorem 2.2.2 of [34] that the original upper embedding of $K_{n,n,n}$ can be reconstructed by lifting this quotient embedding. To do so we need a voltage assignment on $K^m_{n/m,n/m,n/m}$ in $\mathbb{Z}_m$ such that the voltages on the triangles sum to zero while the voltages on the large face sum to an element coprime with m. Since the large face consists of all the edges in the embedding which also form the triangles the voltage sum will always be zero. Contradiction. ■

However, the group can act semi-regularly on two of the parts. The following is an example. In it both m and n are equal to 3 and the cyclic group $\mathbb{Z}_n$ acts semi-regularly on two of the parts and fixes the third.

Example Consider the cyclic Latin square of order 3.

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

An upper embedding of the above Latin square has the following rotation scheme,

$$\begin{array}{lll}
0_r : 0_c 0_e\ 1_c 1_e\ 2_c 2_e & 0_c : 0_e 0_r\ 1_e 1_r\ 2_e 2_r & 0_e : 0_r 0_c\ 1_r 2_c\ 2_r 1_c \\
1_r : 0_c 1_e\ 1_c 2_e\ 2_c 0_e & 1_c : 0_e 2_r\ 1_e 0_r\ 2_e 1_r & 1_e : 0_r 1_c\ 1_r 0_c\ 2_r 2_c \\
2_r : 0_c 2_e\ 1_c 0_e\ 2_c 1_e & 2_c : 0_e 1_r\ 1_e 2_r\ 2_e 0_r & 2_e : 0_r 2_c\ 2_r 0_c\ 1_r 1_c
\end{array}$$

The large face is,

$0_e\ 0_r\ 1_c\ 2_e\ 0_r\ 0_c\ 1_e\ 2_r\ 0_c\ 0_e\ 1_r\ 0_c\ 2_e\ 1_r\ 2_c\ 1_e\ 0_r\ 2_c\ 0_e\ 2_r\ 2_c\ 2_e\ 2_r\ 1_c\ 1_e\ 1_r\ 1_c$

The action of the automorphism group is:

$i_e \mapsto i_e,\ i_r \mapsto (i+1)_r,\ i_c \mapsto (i+2)_c,\ 0 \le i \le 2.$

We can generalize the above example to any cyclic Latin square of odd order n. The rotation scheme and the large face of the upper embedding will be as follows:

$i_r : \ldots\ j_c(i+j)_e\ (j+1)_c(i+j+1)_e\ \ldots$

$j_c : \ldots\ k_e(k-j)_r\ (k+1)_e(k+1-j)_r\ \ldots$

$k_e : \ldots\ i_r(k-i)_c\ (i+1)_r(k-i-1)_c\ \ldots,\ k \ne -1$

$(-1)_e : \ldots\ i_r(-i-1)_c\ (i+t)_r(-i-1-t)_c\ \ldots,\ t \ne 1,\ (t,n)=1,\ (t-1,n)=1$

Such a value of t always exists. For example, we can take $t = 2$.

The large face is,

$$\begin{array}{cccccccccccc}
(-1)_e & t_r & (-t)_c & 1_e & (t+2)_r & (-t)_c & 3_e & (t+4)_r & (-t)_c & \ldots & (-3)_e & (t-2)_r & (-t)_c \\
(-1)_e & (2t-1)_r & (-2t+1)_c & 1_e & (2t+1)_r & (-2t+1)_c & 3_e & (2t+3)_r & (-2t+1)_c & \ldots & (-3)_e & (2t-3)_r & (-2t+1)_c \\
(-1)_e & (3t-2)_r & (-3t+2)_c & 1_e & (3t)_r & (-3t+2)_c & 3_e & (3t+2)_r & (-3t+2)_c & \ldots & (-3)_e & (3t-4)_r & (-3t+2)_c \\
 & \vdots & & & \vdots & & & \vdots & & & & \vdots & \\
(-1)_e & 1_r & (-1)_c & 1_e & 3_r & (-1)_c & 3_e & 5_r & (-1)_c & \ldots & (-3)_e & (-1)_r & (-1)_c
\end{array}$$

The action of the automorphism group is:

$i_e \mapsto i_e,\ i_r \mapsto (i-t)_r,\ i_c \mapsto (i+t),\ 0 \le i \le n-1.$

So to summarize the results so far, we have shown that the group G of orientation-preserving and part preserving automorphisms of an orientable upper embedding of a Latin square of order n is cyclic $\mathbb{Z}_m$ where $m \mid n$. Further the case $m = n$ is achieved. The cyclic group $\mathbb{Z}_m$ cannot act semi-regularly on all three parts of the embedding but can act semi-regularly on two of the three parts. In the construction given, the action of the group $\mathbb{Z}_n$ can be described by the notation $1^n n^1 n^1$ and when $n = p$ is prime this is the only possibility.

Now consider the situation of an automorphism, say ϕ, which fixes one part and interchanges the other two. Then ϕ^2 fixes all three parts. It follows that ϕ has even order. But all automorphisms are of odd order. So we have a contradiction and there are no automorphisms of this type.

Finally consider the situation of an automorphism, say θ, which permutes the parts cyclically. Then θ^3 fixes all three parts. Suppose that θ^3 has an orbit of length i in one part and of length j in a second part where $j < i$. If x is an element of the orbit of length j in the second part then $\theta^{3j}(x) = x$. Further $\theta^{3j}(\theta(x)) = \theta(\theta^{3j}(x)) = \theta(x)$. So θ^{3j} stabilizes vertices in different parts. But θ^{3j} is not the identity because $j < i$. This proves that all orbits of θ^3 have the same length, say m, which must be the order of θ^3. Thus the group generated by θ^3 is semi-regular on all three parts which is a contradiction by Lemma 6.2.4, unless θ^3 is the identity. Hence, any automorphism which permutes the parts cyclically must have order 3.

We assume, without loss of generality, that the automorphism θ which permutes the parts cyclically is of the form $\prod_{i=0}^{n-1}(i_r \ i_c \ i_e)$ since any other automorphism will give a Latin square isotopic to the Latin square obtained by θ. Note that in a Latin square with such an automorphism, if $\{x_r, y_c, z_e\}$ is a triple then $\{x_c, y_e, z_r\}$ and $\{x_e, y_r, z_c\}$ must also be triples. This is equivalent to a Latin square obtained from a quasigroup having the semi-symmetric property, i.e. $xy = z \implies yz = x \implies zx = y$. An example for $n = 5$ is the following.

Example Consider the following Latin square of order 5.

	0	1	2	3	4
0	0	3	4	1	2
1	3	2	1	0	4
2	4	1	3	2	0
3	1	0	2	4	3
4	2	4	0	3	1

An upper embedding of the above Latin square with an automorphism of order 3 which permutes the parts cyclically has the following rotation scheme,

$$
\begin{array}{lll}
0_r : 0_c0_e\ 4_c2_e\ 3_c1_e\ 2_c4_e\ 1_c3_e & 0_c : 0_e0_r\ 4_e2_r\ 3_e1_r\ 2_e4_r\ 1_e3_r & 0_e : 0_r0_c\ 4_r2_c\ 3_r1_c\ 2_r4_c\ 1_r3_c \\
1_r : 0_c3_e\ 4_c4_e\ 3_c0_e\ 1_c2_e\ 2_c1_e & 1_c : 0_e3_r\ 4_e4_r\ 3_e0_r\ 1_e2_r\ 2_e1_r & 1_e : 0_r3_c\ 4_r4_c\ 3_r0_c\ 1_r2_c\ 2_r1_c \\
2_r : 0_c4_e\ 4_c0_e\ 2_c3_e\ 3_c2_e\ 1_c1_e & 2_c : 0_e4_r\ 4_e0_r\ 2_e3_r\ 3_e2_r\ 1_e1_r & 2_e : 0_r4_c\ 4_r0_c\ 2_r3_c\ 3_r2_c\ 1_r1_c \\
3_r : 0_c1_e\ 4_c3_e\ 2_c2_e\ 1_c0_e\ 3_c4_e & 3_c : 0_e1_r\ 4_e3_r\ 2_e2_r\ 1_e0_r\ 3_e4_r & 3_e : 0_r1_c\ 4_r3_c\ 2_r2_c\ 1_r0_c\ 3_r4_c \\
4_r : 0_c2_e\ 4_c1_e\ 3_c3_e\ 2_c0_e\ 1_c4_e & 4_c : 0_e2_r\ 4_e1_r\ 3_e3_r\ 2_e0_r\ 1_e4_r & 4_e : 0_r2_c\ 4_r1_c\ 3_r3_c\ 2_r0_c\ 1_r4_c
\end{array}
$$

The large face is,

$0_e\ 0_r\ 4_c\ 1_e\ 3_r\ 4_c\ 2_e\ 4_r\ 4_c\ 0_e\ 1_r\ 1_c\ 0_e\ 2_r\ 2_c\ 1_e\ 2_r\ 0_c\ 3_e\ 3_r\ 2_c\ 3_e\ 1_r\ 4_c\ 3_e\ 0_r\ 0_c\ 4_e\ 1_r\ 3_c$
$4_e\ 2_r\ 4_c\ 4_e\ 0_r\ 1_c\ 1_e\ 0_r\ 2_c\ 2_e\ 1_r\ 2_c\ 0_e\ 3_r\ 3_c\ 2_e\ 3_r\ 1_c\ 4_e\ 3_r\ 0_c\ 0_e\ 4_r\ 1_c\ 3_e\ 4_r\ 2_c\ 4_e\ 4_r\ 0_c$
$1_e\ 1_r\ 0_c\ 2_e\ 2_r\ 1_c\ 2_e\ 0_r\ 3_c\ 3_e\ 2_r\ 3_c\ 1_e\ 4_r\ 3_c\ 0_e\ 0_r$

If the Latin square is also idempotent, i.e. $xx = x$, the quasigroup corresponds to a Mendelsohn triple system (MTS). There are, up to isomorphism, three Mendelsohn triple systems of order 7 [7]. Upper embeddings of each of these with an orientation-preserving automorphism which permutes the parts cyclically are below.

Example The three nonisomorphic MTS(7) on base set $\mathbb{Z}_7$ are:

1. $[\{0,1,3\}\ \{0,3,1\}]\ (\text{mod } 7)$,
2. $[\{0,1,3\}\ \{0,3,2\}]\ (\text{mod } 7)$,
3. $[\{0,1,2\}, \{0,2,1\}, \{0,3,4\}, \{0,4,3\}, \{0,5,6\}, \{0,6,5\}, \{1,3,5\}, \{1,6,3\}, \{1,5,4\}, \{1,4,6\}, \{2,5,3\}, \{2,3,6\}, \{2,4,5\}, \{3,6,4\}]$.

For each of the following three examples, the Latin square corresponds to the respective Mendelsohn triple system. Each Latin square is followed by its upper embedding rotation scheme and large face. It is easy to see that each upper embedding has an orientation-preserving automorphism which permutes the parts cyclically,

1. $[\{0,1,3\}\ \{0,3,1\}]\ (\mathrm{mod}\ 7)$

	0	1	2	3	4	5	6
0	0	3	6	1	5	4	2
1	3	1	4	0	2	6	5
2	6	4	2	5	1	3	0
3	1	0	5	3	6	2	4
4	5	2	1	6	4	0	3
5	4	6	3	2	0	5	1
6	2	5	0	4	3	1	6

$0_r : 0_c0_e\ 6_c2_e\ 5_c4_e\ 4_c5_e\ 3_c1_e\ 2_c6_e\ 1_c3_e$

$1_r : 1_c1_e\ 0_c3_e\ 6_c5_e\ 5_c6_e\ 4_c2_e\ 3_c0_e\ 2_c4_e$

$2_r : 2_c2_e\ 1_c4_e\ 0_c6_e\ 6_c0_e\ 5_c3_e\ 4_c1_e\ 3_c5_e$

$3_r : 3_c3_e\ 2_c5_e\ 1_c0_e\ 0_c1_e\ 6_c4_e\ 5_c2_e\ 4_c6_e$

$4_r : 4_c4_e\ 3_c6_e\ 2_c1_e\ 1_c2_e\ 0_c5_e\ 6_c3_e\ 5_c0_e$

$5_r : 5_c5_e\ 4_c0_e\ 3_c2_e\ 2_c3_e\ 1_c6_e\ 0_c4_e\ 6_c1_e$

$6_r : 6_c6_e\ 0_c2_e\ 1_c5_e\ 2_c0_e\ 3_c4_e\ 4_c3_e\ 5_c1_e$

$0_c : 0_e0_r\ 6_e2_r\ 5_e4_r\ 4_e5_r\ 3_e1_r\ 2_e6_r\ 1_e3_r$

$1_c : 1_e1_r\ 0_e3_r\ 6_e5_r\ 5_e6_r\ 4_e2_r\ 3_e0_r\ 2_e4_r$

$2_c : 2_e2_r\ 1_e4_r\ 0_e6_r\ 6_e0_r\ 5_e3_r\ 4_e1_r\ 3_e5_r$

$3_c : 3_e3_r\ 2_e5_r\ 1_e0_r\ 0_e1_r\ 6_e4_r\ 5_e2_r\ 4_e6_r$

$4_c : 4_e4_r\ 3_e6_r\ 2_e1_r\ 1_e2_r\ 0_e5_r\ 6_e3_r\ 5_e0_r$

$5_c : 5_e5_r\ 4_e0_r\ 3_e2_r\ 2_e3_r\ 1_e6_r\ 0_e4_r\ 6_e1_r$

$6_c : 6_e6_r\ 0_e2_r\ 1_e5_r\ 2_e0_r\ 3_e4_r\ 4_e3_r\ 5_e1_r$

$0_e : 0_r0_c\ 6_r2_c\ 5_r4_c\ 4_r5_c\ 3_r1_c\ 2_r6_c\ 1_r3_c$

$1_e : 1_r1_c\ 0_r3_c\ 6_r5_c\ 5_r6_c\ 4_r2_c\ 3_r0_c\ 2_r4_c$

$2_e : 2_r2_c\ 1_r4_c\ 0_r6_c\ 6_r0_c\ 5_r3_c\ 4_r1_c\ 3_r5_c$

$3_e : 3_r3_c\ 2_r5_c\ 1_r0_c\ 0_r1_c\ 6_r4_c\ 5_r2_c\ 4_r6_c$

$4_e : 4_r4_c\ 3_r6_c\ 2_r1_c\ 1_r2_c\ 0_r5_c\ 6_r3_c\ 5_r0_c$

$5_e : 5_r5_c\ 4_r0_c\ 3_r2_c\ 2_r3_c\ 1_r6_c\ 0_r4_c\ 6_r1_c$

$6_e : 6_r6_c\ 0_r2_c\ 1_r5_c\ 2_r0_c\ 3_r4_c\ 4_r3_c\ 5_r1_c$

$0_e\ 0_r\ 6_c\ 3_e\ 3_r\ 2_c\ 4_e\ 0_r\ 4_c\ 4_e\ 3_r\ 5_c\ 1_e\ 5_r\ 5_c\ 4_e\ 6_r\ 4_c\ 2_e\ 0_r\ 5_c\ 3_e\ 1_r\ 6_c\ 6_e\ 0_r\ 1_c\ 2_e\ 3_r\ 4_c$
$5_e\ 6_r\ 2_c\ 6_e\ 1_r\ 4_c\ 1_e\ 1_r\ 0_c\ 2_e\ 5_r\ 2_c\ 2_e\ 1_r\ 3_c\ 6_e\ 5_r\ 0_c\ 3_e\ 0_r\ 0_c\ 6_e\ 3_r\ 3_c\ 2_e\ 4_r\ 0_c\ 4_e\ 4_r\ 3_c$
$5_e\ 1_r\ 5_c\ 5_e\ 4_r\ 6_c\ 4_e\ 2_r\ 0_c\ 5_e\ 3_r\ 1_c\ 6_e\ 6_r\ 0_c\ 1_e\ 2_r\ 3_c\ 4_e\ 5_r\ 6_c\ 2_e\ 6_r\ 1_c\ 4_e\ 1_r\ 1_c\ 0_e\ 2_r\ 5_c$
$2_e\ 2_r\ 1_c\ 3_e\ 6_r\ 5_c\ 0_e\ 3_r\ 0_c\ 0_e\ 6_r\ 3_c\ 3_e\ 2_r\ 4_c\ 0_e\ 4_r\ 4_c\ 3_e\ 5_r\ 1_c\ 5_e\ 5_r\ 4_c\ 6_e\ 4_r\ 2_c\ 0_e\ 5_r\ 3_c$
$1_e\ 6_r\ 6_c\ 0_e\ 1_r\ 2_c\ 3_e\ 4_r\ 5_c\ 6_e\ 2_r\ 6_c\ 1_e\ 4_r\ 1_c\ 1_e\ 0_r\ 2_c\ 5_e\ 2_r\ 2_c\ 1_e\ 3_r\ 6_c\ 5_e\ 0_r\ 3_c$

2. $[\{0,1,3\}\ \{0,3,2\}]$ (mod 7)

	0	1	2	3	4	5	6
0	0	3	6	2	5	1	4
1	5	1	4	0	3	6	2
2	3	6	2	5	1	4	0
3	1	4	0	3	6	2	5
4	6	2	5	1	4	0	3
5	4	0	3	6	2	5	1
6	2	5	1	4	0	3	6

$0_r : 0_c0_e\ 6_c4_e\ 5_c1_e\ 4_c5_e\ 3_c2_e\ 2_c6_e\ 1_c3_e$

$1_r : 1_c1_e\ 0_c5_e\ 6_c2_e\ 5_c6_e\ 4_c3_e\ 3_c0_e\ 2_c4_e$

$2_r : 2_c2_e\ 1_c6_e\ 0_c3_e\ 6_c0_e\ 5_c4_e\ 4_c1_e\ 3_c5_e$

$3_r : 3_c3_e\ 2_c0_e\ 1_c4_e\ 0_c1_e\ 6_c5_e\ 5_c2_e\ 4_c6_e$

$4_r : 4_c4_e\ 3_c1_e\ 2_c5_e\ 1_c2_e\ 0_c6_e\ 6_c3_e\ 5_c0_e$

$5_r : 5_c5_e\ 4_c2_e\ 3_c6_e\ 2_c3_e\ 1_c0_e\ 0_c4_e\ 6_c1_e$

$6_r : 6_c6_e\ 1_c5_e\ 2_c1_e\ 4_c0_e\ 0_c2_e\ 5_c3_e\ 3_c4_e$

$0_c : 0_e0_r\ 6_e4_r\ 5_e1_r\ 4_e5_r\ 3_e2_r\ 2_e6_r\ 1_e3_r$

$1_c : 1_e1_r\ 0_e5_r\ 6_e2_r\ 5_e6_r\ 4_e3_r\ 3_e0_r\ 2_e4_r$

$2_c : 2_e2_r\ 1_e6_r\ 0_e3_r\ 6_e0_r\ 5_e4_r\ 4_e1_r\ 3_e5_r$

$3_c : 3_e3_r\ 2_e0_r\ 1_e4_r\ 0_e1_r\ 6_e5_r\ 5_e2_r\ 4_e6_r$

$4_c : 4_e4_r\ 3_e1_r\ 2_e5_r\ 1_e2_r\ 0_e6_r\ 6_e3_r\ 5_e0_r$

$5_c : 5_e5_r\ 4_e2_r\ 3_e6_r\ 2_e3_r\ 1_e0_r\ 0_e4_r\ 6_e1_r$

$6_c : 6_e6_r\ 1_e5_r\ 2_e1_r\ 4_e0_r\ 0_e2_r\ 5_e3_r\ 3_e4_r$

$0_e : 0_r0_c\ 6_r4_c\ 5_r1_c\ 4_r5_c\ 3_r2_c\ 2_r6_c\ 1_r3_c$

$1_e : 1_r1_c\ 0_r5_c\ 6_r2_c\ 5_r6_c\ 4_r3_c\ 3_r0_c\ 2_r4_c$

$2_e : 2_r2_c\ 1_r6_c\ 0_r3_c\ 6_r0_c\ 5_r4_c\ 4_r1_c\ 3_r5_c$

$3_e : 3_r3_c\ 2_r0_c\ 1_r4_c\ 0_r1_c\ 6_r5_c\ 5_r2_c\ 4_r6_c$

$4_e : 4_r4_c\ 3_r1_c\ 2_r5_c\ 1_r2_c\ 0_r6_c\ 6_r3_c\ 5_r0_c$

$5_e : 5_r5_c\ 4_r2_c\ 3_r6_c\ 2_r3_c\ 1_r0_c\ 0_r4_c\ 6_r1_c$

$6_e : 6_r6_c\ 1_r5_c\ 2_r1_c\ 4_r0_c\ 0_r2_c\ 5_r3_c\ 3_r4_c$

$0_e\ 0_r\ 6_c\ 0_e\ 1_r\ 2_c\ 3_e\ 4_r\ 5_c\ 6_e\ 2_r\ 0_c\ 2_e\ 5_r\ 3_c\ 5_e\ 1_r\ 6_c\ 4_e\ 6_r\ 6_c\ 1_e\ 4_r\ 2_c\ 4_e\ 0_r\ 5_c\ 0_e\ 3_r\ 1_c$
$3_e\ 6_r\ 3_c\ 3_e\ 2_r\ 6_c\ 5_e\ 2_r\ 2_c\ 1_e\ 5_r\ 5_c\ 4_e\ 1_r\ 1_c\ 0_e\ 4_r\ 4_c\ 3_e\ 0_r\ 0_c\ 6_e\ 0_r\ 1_c\ 2_e\ 3_r\ 4_c\ 5_e\ 6_r\ 2_c$
$0_e\ 2_r\ 5_c\ 3_e\ 5_r\ 1_c\ 6_e\ 4_r\ 6_c\ 6_e\ 1_r\ 4_c\ 2_e\ 4_r\ 0_c\ 5_e\ 0_r\ 3_c\ 1_e\ 3_r\ 6_c\ 3_e\ 3_r\ 2_c\ 6_e\ 5_r\ 2_c\ 2_e\ 1_r\ 5_c$
$5_e\ 4_r\ 1_c\ 1_e\ 0_r\ 4_c\ 4_e\ 3_r\ 0_c\ 0_e\ 6_r\ 0_c\ 1_e\ 2_r\ 3_c\ 4_e\ 5_r\ 6_c\ 2_e\ 0_r\ 2_c\ 5_e\ 3_r\ 5_c\ 1_e\ 6_r\ 4_c\ 6_e\ 6_r\ 1_c$
$4_e\ 2_r\ 4_c\ 0_e\ 5_r\ 0_c\ 3_e\ 1_r\ 3_c\ 6_e\ 3_r\ 3_c\ 2_e\ 6_r\ 5_c\ 2_e\ 2_r\ 1_c\ 5_e\ 5_r\ 4_c\ 1_e\ 1_r\ 0_c\ 4_e\ 4_r\ 3_c$

3. [{0, 1, 2}, {0, 2, 1}, {0, 3, 4}, {0, 4, 3}, {0, 5, 6}, {0, 6, 5}, {1, 3, 5}, {1, 6, 3}, {1, 5, 4}, {1, 4, 6}, {2, 5, 3}, {2, 3, 6}, {2, 4, 5}, {3, 6, 4}]

	0	1	2	3	4	5	6
0	0	2	1	4	3	6	5
1	2	1	0	5	6	4	3
2	1	0	2	6	5	3	4
3	4	6	5	3	0	1	2
4	3	5	6	0	4	2	1
5	6	3	4	2	1	5	0
6	5	4	3	1	2	0	6

$0_r : 0_c0_e\ 6_c5_e\ 4_c3_e\ 2_c1_e\ 1_c2_e\ 5_c6_e\ 3_c4_e$

$1_r : 1_c1_e\ 0_c6_e\ 5_c4_e\ 3_c2_e\ 2_c3_e\ 6_c0_e\ 4_c5_e$

$2_r : 2_c2_e\ 1_c0_e\ 6_c5_e\ 4_c3_e\ 3_c4_e\ 0_c1_e\ 5_c6_e$

$3_r : 3_c3_e\ 2_c1_e\ 0_c6_e\ 5_c4_e\ 4_c5_e\ 1_c2_e\ 6_c0_e$

$4_r : 4_c4_e\ 3_c2_e\ 1_c0_e\ 6_c5_e\ 5_c6_e\ 2_c3_e\ 0_c1_e$

$5_r : 5_c5_e\ 4_c3_e\ 2_c1_e\ 0_c6_e\ 6_c0_e\ 3_c4_e\ 1_c2_e$

$6_r : 6_c6_e\ 2_c3_e\ 4_c5_e\ 1_c0_e\ 0_c1_e\ 3_c2_e\ 5_c4_e$

$0_c : 0_e0_r\ 6_e5_r\ 4_e3_r\ 2_e1_r\ 1_e2_r\ 5_e6_r\ 3_e4_r$

$1_c : 1_e1_r\ 0_e6_r\ 5_e4_r\ 3_e2_r\ 2_e3_r\ 6_e0_r\ 4_e5_r$

$2_c : 2_e2_r\ 1_e0_r\ 6_e5_r\ 4_e3_r\ 3_e4_r\ 0_e1_r\ 5_e6_r$

$3_c : 3_e3_r\ 2_e1_r\ 0_e6_r\ 5_e4_r\ 4_e5_r\ 1_e2_r\ 6_e0_r$

$4_c : 4_e4_r\ 3_e2_r\ 1_e0_r\ 6_e5_r\ 5_e6_r\ 2_e3_r\ 0_e1_r$

$5_c : 5_e5_r\ 4_e3_r\ 2_e1_r\ 0_e6_r\ 6_e0_r\ 3_e4_r\ 1_e2_r$

$6_c : 6_e6_r\ 2_e3_r\ 4_e5_r\ 1_e0_r\ 0_e1_r\ 3_e2_r\ 5_e4_r$

$0_e : 0_r0_c\ 6_r5_c\ 4_r3_c\ 2_r1_c\ 1_r2_c\ 5_r6_c\ 3_r4_c$

$1_e : 1_r1_c\ 0_r6_c\ 5_r4_c\ 3_r2_c\ 2_r3_c\ 6_r0_c\ 4_r5_c$

$2_e : 2_r2_c\ 1_r0_c\ 6_r5_c\ 4_r3_c\ 3_r4_c\ 0_r1_c\ 5_r6_c$

$3_e : 3_r3_c\ 2_r1_c\ 0_r6_c\ 5_r4_c\ 4_r5_c\ 1_r2_c\ 6_r0_c$

$4_e : 4_r4_c\ 3_r2_c\ 1_r0_c\ 6_r5_c\ 5_r6_c\ 2_r3_c\ 0_r1_c$

$5_e : 5_r5_c\ 4_r3_c\ 2_r1_c\ 0_r6_c\ 6_r0_c\ 3_r4_c\ 1_r2_c$

$6_e : 6_r6_c\ 2_r3_c\ 4_r5_c\ 1_r0_c\ 0_r1_c\ 3_r2_c\ 5_r4_c$

$0_e\ 0_r\ 6_c\ 0_e\ 3_r\ 3_c\ 2_e\ 3_r\ 6_c\ 4_e\ 2_r\ 0_c\ 5_e\ 3_r\ 1_c\ 6_e\ 3_r\ 5_c\ 2_e\ 4_r\ 1_c\ 3_e\ 0_r\ 2_c\ 6_e\ 5_r\ 6_c\ 1_e\ 5_r\ 0_c$
$4_e\ 6_r\ 6_c\ 2_e\ 2_r\ 1_c\ 2_e\ 5_r\ 5_c\ 4_e\ 5_r\ 1_c\ 1_e\ 0_r\ 1_c\ 4_e\ 4_r\ 3_c\ 4_e\ 0_r\ 0_c\ 6_e\ 0_r\ 3_c\ 3_e\ 2_r\ 3_c\ 6_e\ 4_r\ 2_c$
$0_e\ 5_r\ 3_c\ 1_e\ 6_r\ 3_c\ 5_e\ 2_r\ 4_c\ 1_e\ 3_r\ 0_c\ 2_e\ 6_r\ 5_c\ 6_e\ 1_r\ 5_c\ 0_e\ 4_r\ 6_c\ 6_e\ 2_r\ 2_c\ 1_e\ 2_r\ 5_c\ 5_e\ 4_r\ 5_c$
$1_e\ 1_r\ 0_c\ 1_e\ 4_r\ 4_c\ 3_e\ 4_r\ 0_c\ 0_e\ 6_r\ 0_c\ 3_e\ 3_r\ 2_c\ 3_e\ 6_r\ 4_c\ 2_e\ 0_r\ 5_c\ 3_e\ 1_r\ 6_c\ 3_e\ 5_r\ 2_c\ 4_e\ 1_r\ 3_c$
$0_e\ 2_r\ 6_c\ 5_e\ 6_r\ 1_c\ 5_e\ 0_r\ 4_c\ 6_e\ 6_r\ 2_c\ 2_e\ 1_r\ 2_c\ 5_e\ 5_r\ 4_c\ 5_e\ 1_r\ 1_c\ 0_e\ 1_r\ 4_c\ 4_e\ 3_r\ 4_c$

CHAPTER 7

Graphs in Steiner triple systems

In this chapter we investigate when a graph can be represented in a Steiner triple system. We say that a Steiner triple system $T = (P, B)$ *represents* a graph $G = (V, E)$ if there exists a one-to-one function $\phi : V(G) \to P(T)$ such that the induced function $\phi : E(G) \to B(T)$ is also one-to-one. In other words, if every edge $e = \{u, v\}$ of the graph has its image $\{\phi(u), \phi(v)\}$ in a distinct block of the Steiner triple system then the graph is representable in the Steiner triple system. We do not allow loops or multiple edges but the graph may be disconnected.

Example Let G be the 7-cycle $(0, 1, 2, 3, 4, 5, 6)$ in $\mathbb{Z}_7$. Let T be the Fano plane whose points are also the elements of $\mathbb{Z}_7$ and whose blocks are cyclic shifts of a starter block $\{0, 1, 3\}$. Mapping vertex i to point i gives a representation of G in T.

In the next section we explain how this question of finding representations of graphs in Steiner triple systems is closely related to finding independent sets in Steiner triple systems. Indeed our question is a generalization of finding such independent sets. In addition, we give a bound which ensures that every graph of order n is represented in some STS(m) and a bound which ensures that every graph of order n is represented in every STS(m).

7.1 Independent sets in Steiner triple systems

An *independent set* of a Steiner triple system $T = (P, B)$ is a subset U of P such that no three points of U occur in a single block of B. Therefore, in order to represent K_n, an STS(m) with an independent set of cardinality n is required. It is easy to see that such an STS(m) represents any graph of order n. We can state the following,

Lemma 7.1.1 *If a Steiner triple system T represents a graph G, then T represents any subgraph H of G.*

Proof The same $\phi : V(G) \to P(T)$ representing G also represents H. ∎

We now answer the following questions:

1. Determine $f(n)$ such that there exists a Steiner triple system of order $f(n)$ which represents every graph of order n, and

2. Determine $g(n)$ such that every Steiner triple system of order $g(n)$ represents every graph of order n.

Denote the maximal independent set over all STS(m)s by $\beta_{max}(m)$ and the smallest maximum independent set over all STS(m)s by $\beta_{min}(m)$. To answer question 1 it suffices to find the smallest order m of a Steiner triple system with $\beta_{max}(m) \geq n$. This was done by Sauer and Schönheim [53].

Proposition 7.1.2 *The size of the maximal independent set in any Steiner triple system of order m is*

$$\beta_{max}(m) = \begin{cases} (m+1)/2 & \text{if } m \equiv 3, 7 \pmod{12} \\ (m-1)/2 & \text{if } m \equiv 1, 9 \pmod{12}. \end{cases}$$

Theorem 7.1.3 *There is a Steiner triple system of every order* $m \geq f(n)$ *that represents every graph of order* n *where*

$$f(n) = \begin{cases} 2n-1 & n \equiv 2,4 \pmod 6 \\ 2n+1 & n \equiv 0,1,3 \pmod 6 \\ 2n+3 & n \equiv 5 \pmod 6. \end{cases}$$

Proof We want to find the smallest order m of a Steiner triple system with $\beta_{max}(m) \geq n$. Therefore, from Proposition 7.1.2 we have $m_1 \geq 2n-1$ where $m_1 \equiv 3, 7 \pmod{12}$ and $m_2 \geq 2n+1$ where $m_2 \equiv 1, 9 \pmod{12}$.

If $n \equiv 0 \pmod 6$, then $m_1 \geq 12s - 1$ and $m_2 \geq 12s + 1$ where $s \geq 0$. Clearly in this case, the smallest admissible order of a Steiner triple system is m_2. Thus, for $n \equiv 0 \pmod 6$, $f(n) = 2n+1$. Similarly, for $n \equiv 2, 4 \pmod 6$, $f(n) = 2n-1$. If $n \equiv 1, 3 \pmod 6$, the smallest order of m will be the same as for $n+1 \equiv 2, 4 \pmod 6$, i.e. $f(n) = 2(n+1) - 1 = 2n+1$. Finally, if $n \equiv 5 \pmod 6$, the smallest order of m will be the same as for $n+1 \equiv 0 \pmod 6$, i.e. $f(n) = 2(n+1) + 1 = 2n+3$. ∎

To answer question 2 we need $\beta_{min}(m)$. The following was proven, using a probabilistic argument, by Phelps and Rödl [50].

Proposition 7.1.4 *There exists an absolute constant* $c > 0$ *such that every Steiner triple system of order* m *has an independent set of size* $n \geq c\sqrt{m \log m}$.

As a consequence, the inverse of the function in the previous proposition is the desired $g(n)$. However, this inverse cannot be expressed using elementary functions. Therefore, to obtain an explicit $g(n)$ we can use a weaker result, by Erdös and Hajnal [19], with a much simpler non-probabilistic proof, see [8], page 305.

Proposition 7.1.5 *Every Steiner triple system of order* m *has an independent set of size at least* $\lfloor \sqrt{2m} \rfloor$.

Theorem 7.1.6 *Every Steiner triple system of order $m \geq g(n)$ represents every graph of order n where*

$$g(n) = \begin{cases} (n^2+1)/2 & n \equiv 1,5 \pmod 6 \\ (n^2+2)/2 & n \equiv 0,2,4 \pmod 6 \\ (n^2+5)/2 & n \equiv 3 \pmod 6. \end{cases}$$

Proof From Proposition 7.1.5 we have $m \geq n^2/2$. The table below gives the smallest order m such that every STS(m) can represent a graph of the corresponding order n.

n	$n^2/2$	m
$6s$	$18s^2$	$n^2/2+1$
$6s+1$	$18s^2+6s+1/2$	$n^2/2+1/2$
$6s+2$	$18s^2+12s+2$	$n^2/2+1$
$6s+3$	$18s^2+18s+9/2$	$n^2/2+5/2$
$6s+4$	$18s^2+24s+8$	$n^2/2+1$
$6s+5$	$18s^2+30s+25/2$	$n^2/2+1/2$

■

7.2 Small order Steiner triple systems

In this section we investigate representations of graphs in Steiner triple systems of small order. A graph may only be represented in an STS(m) if it has at most m vertices and $m(m-1)/6$ edges. The following lemma gives some other necessary conditions.

Lemma 7.2.1 *If G has either of the following properties, then it cannot be represented in any $STS(m)$:*

1. *two non-adjacent vertices of degree $(m-1)/2$,*

2. *two non-adjacent vertices of degree* $(m-3)/2$ *with common neighbours in a graph with* $m(m-1)/6$ *edges.*

Proof Suppose that G has two non-adjacent vertices, u and v, of degree $(m-1)/2$. The $(m-1)/2$ blocks through $\phi(u)$ represent $(m-1)/2$ edges and together contain all m points of the Steiner triple system. The same is true for $\phi(v)$. Now the block containing $\phi(u)$ and $\phi(v)$ either represents two separate edges, or an edge joining u and v. Both are contradictions.

Now suppose that G has two non-adjacent vertices, u and v, of degree $(m-3)/2$. Let z be the third point in the block containing $\phi(u)$ and $\phi(v)$ and $B = \{\phi(u), \phi(v), z\}$. If z represents one of u, v's common neighbours, then B represents two edges, a contradiction. However, B cannot represent the non-existent edge $\{u, v\}$, so it represents no edge at all. Since G has $m(m-1)/6$ edges, it cannot be represented. ■

If we have a graph represented in an STS(m) and it has fewer than m vertices then isolated vertices can always be added to the representation. If it has fewer than $m(m-1)/6$ edges, unused blocks can then be used to add edges in the representation. After applying this procedure the graph has at most one isolated vertex; if there were two isolated vertices u, v then the block containing the points $\phi(u)$, $\phi(v)$ could represent no edge. It will be convenient to ignore an isolated vertex, if it exists, and therefore say that a graph representable in an STS(m) is *maximal* if it has m or $m-1$ vertices and $m(m-1)/6$ edges.

A non-representable graph in an STS(m) is said to be *minimal* if by removing any of its edges or any of its vertices gives a new graph which is representable. The complete set of minimal graphs, $\mathbb{G}^m_{min}$, that cannot be represented in an STS(m) is a set of *obstructions*: G cannot be represented if and only if it contains a subgraph $H \in \mathbb{G}^m_{min}$. Any such obstruction has at most $m+1$ vertices and at most $m(m-1)/6+1$ edges.

Proposition 7.2.2 *There is exactly one maximal graph that can be represented and two minimal graphs that cannot be represented in the STS*(3)*. These are given in Figure 7.1 below, using the Steiner triple system* $\{0, 1, 2\}$.

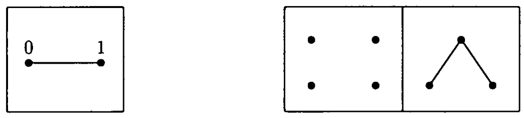


Figure 7.1: The maximal and minimal graphs in the STS(3).

Proposition 7.2.3 *There are exactly 16 maximal graphs that can be represented in the Fano plane. These are illustrated in Figure 7.2.*

Proof A complete catalogue of graphs with 7 vertices and 7 edges is given in [51], pages 13 and 14. In what follows the references are to this listing. There are 34 graphs with maximum degree 3, one of which (G293) has two isolated points and can be eliminated, ten of which (G298 to G299, G305 to G308, G310 to G313) have an isolated point and twenty-three of which (G327 to G330, G336 to G354) are connected. Of these, G298, G305, G308, G311, G327, G329, G341 to G343, and G347 cannot be represented using criterion 1 of Lemma 7.2.1, and G308, G311, G341, G344, G346 to G347, G350, and G354 cannot be represented using criterion 2 of Lemma 7.2.1.

This leaves 19 graphs to consider. The graphs G312, G337, and G348 can also be eliminated since each one contains a minimal obstruction as a subgraph. All minimal obstructions are given in Proposition 7.2.4 below. The remaining 16 graphs, G299, G306 to G307, G310, G313, G328, G330, G336, G338 to G340, G345, G349, and G351 to G353 can be represented. These are illustrated in Figure 7.2 below, using the Steiner triple system on elements of $\mathbb{Z}_7$, obtained by cyclic shifts of $\{0, 1, 3\}$. ■

Proposition 7.2.4 *There are exactly 8 minimal graphs that cannot be represented in the Fano plane. These are illustrated in Figure 7.3.*

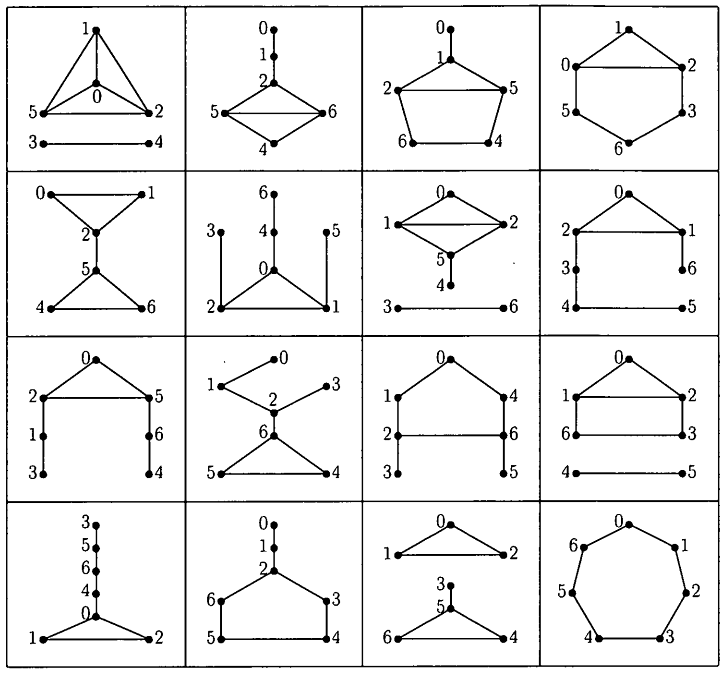


Figure 7.2: The 16 maximal Fano planar graphs.

Proof The first of these graphs is on 8 vertices with no edges, $\bar{K}_8$, and the next is the star $K_{1,4}$. Each of these is obvious: there are too many vertices or a vertex of too high a degree. Any other minimal graph must have

(i) 7 or less vertices,

(ii) 8 or less edges,

(iii) no vertex of valency greater than 3.

Consider first the graphs with 5 edges. There is one on 4 vertices (G17), five on 5 vertices (G34 to G38), nine on 6 vertices (G77 to G85) and six on 7 vertices (G243 to G248). Some of these have a vertex whose valency is greater than 3 and so can be eliminated. All others are representable; they are subgraphs of the 16 maximal Fano planar graphs.

Now consider graphs with 6 edges. There is one on 4 vertices which is K_4 (G18), five on 5 vertices (G40 to G44), fifteen on 6 vertices (G92 to G106) and twenty on 7 vertices (G270 to G289). Most of these graphs are either subgraphs of the 16 maximal Fano planar graphs, which makes them representable, or have a vertex of degree greater than 3. However, five of them cannot be represented and are minimal obstructions; these are G44, G98, G99, G278 and G288.

Next consider the graphs with 7 edges. The only one on 5 or less vertices with no vertex of valency greater than 3 is G48 which is not representable but is not minimal because it contains $K_{2,3}$ (G44). On 6 vertices, there are twenty graphs (G111 to G130) half of which have a vertex of valency greater than 3 and five of which can be represented. The remaining five graphs cannot be represented but are not minimal because they contain a minimal obstruction with 6 edges. Finally, on 7 vertices, there are forty-one graphs (G314 to G354). Eighteen of them have a vertex of valency greater than 3 and eleven of them are subgraphs of the maximal Fano graphs. This leaves twelve graphs eleven of which cannot be represented and contain a minimal obstruction with 6 edges. Therefore, the remaining graph G348 is the only minimal obstruction with 7 edges.

Finally, consider the graphs with 8 edges. There are two on 5 or less vertices (G49, G50), twenty-two on 6 vertices (G133 to G154) and seventy three on 7 vertices (G379 to G451). Each of these graphs contains at least one of the minimal obstructions found above.

Consequently, there are 8 minimal obstructions. We've already stated why the first two, $\bar{K}_8$ and $K_{1,4}$, cannot be represented. We will now give a short proof for each of the remaining six obstructions. Let the STS(7) be defined on the set $\{0, 1, 2, 3, 4, 5, 6\}$.

In Figure 7.3 below, the third graph is $K_{2,3}$ (G44), the fifth graph (G99) is a 4-cycle with 2 non-adjacent pendant edges, and the seventh graph (G278) is a path of length 4 with pendant edges from the 2nd and 4th vertices. All these graphs have two non-adjacent vertices of degree 3 and so cannot be represented by criterion 1 of Lemma 7.2.1.

The fourth graph (G98) is a 4-cycle with 2 non-adjacent pendant vertices at a distance 3 between them. First consider the 4-cycle and represent it by $(0, 1, 2, 3)$. Then the six pairs $\{0, 1\}$, $\{0, 2\}$, $\{0, 3\}$, $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ have to appear in six distinct blocks, forcing the seventh block to be $\{4,5,6\}$. The remaining two edges of the original graph must be represented by the blocks containing the pairs $\{0,2\}$ and $\{1,3\}$. However, the blocks containing these pairs will have a common third point $x \in \{4,5,6\}$ which makes the representation impossible.

The sixth graph (G288) is a 4-cycle disjoint from a path of length 2. Let the 4-cycle be represented by $(0,1,2,3)$. Then the six pairs $\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$ have to appear in six distinct blocks. This forces the seventh block of the system to be $\{4,5,6\}$, which makes the representation of a path of length 2 impossible.

The last graph (G348) is a 6-cycle with a pendant edge. Represent the 6-cycle by $(1,2,3,4,5,6)$. Then the blocks containing the pairs $\{1,2\}$ and $\{3,4\}$ will have a common third point x which cannot be either 5 or 6. Therefore, $x = 0$ and the blocks containing 0 are $\{0,1,2\}$, $\{0,3,4\}$, and $\{0,5,6\}$. To get a representation of the original graph, an edge $(0,y)$, where y is a vertex in the cycle, has to be added. But all three blocks through 0 have been used. Thus, it cannot be represented. ∎

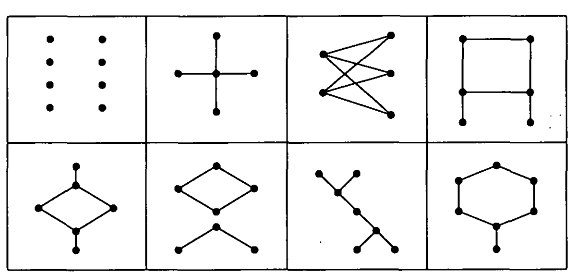
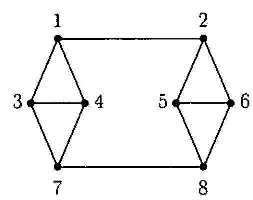


Figure 7.3: The 8 minimal non-Fano-planar graphs.

The above calculations for the STS(7) are infeasible for the STS(9). Considering only connected graphs, there are 4495 graphs on 9 vertices with 12 edges and 1169 graphs on 8 vertices with 12 edges, see [51], page 7. But there are just 5 connected regular graphs of valency 3 on 8 vertices. These are C4 to C8 given in [51], page 127. There is also one disconnected graph consisting of two copies of the complete graph K_4. We find that C5, C6, and C8 cannot be represented.

For C5, the reason is that it contains two non-adjacent degree-3 vertices with common neighbours; criterion 2 in Lemma 7.2.1.

The argument to rule out C6 is more involved but straightforward. Without loss of generality let the blocks through the point 0 be $\{0,1,2\}$, $\{0,3,4\}$, $\{0,5,6\}$, $\{0,7,8\}$ and assume the graph is represented in the following way,



Consider the blocks through 1, $\{1,3,\alpha\}$ and $\{1,4,\beta\}$, that represent the edges $\{1,3\}$ and $\{1,4\}$ respectively. Note that α (resp. β) cannot be 4 (resp. 3) since the pair $\{3,4\}$ is already in a block. Moreover, α (resp. β) cannot be 7 since the edges $\{1,3\}$ and $\{3,7\}$ (resp. $\{1,4\}$ and $\{4,7\}$) have to be represented in different blocks. Therefore, the remaining three blocks through 1 can be completed in one of the following ways:

(i) $\{1,3,5\},\{1,4,6\},\{1,7,8\}$

(ii) $\{1,3,5\},\{1,4,8\},\{1,6,7\}$

(iii) $\{1,3,6\},\{1,4,5\},\{1,7,8\}$

(iv) $\{1,3,6\},\{1,4,8\},\{1,5,7\}$

(v) $\{1,3,8\},\{1,4,5\},\{1,6,7\}$

(vi) $\{1,3,8\},\{1,4,6\},\{1,5,7\}$

But $\{1,7,8\}$ cannot be a block since there exists a block $\{0,7,8\}$ in the system, so (i) and (iii) can be discarded. Furthermore, the blocks $\{1,5,7\}$ and $\{1,6,7\}$ cannot represent any edge; the representation is thus impossible. Contradiction.

Graph C8 is bipartite. Let the bipartition be $\{A_1,A_2,A_3,A_4\}$, $\{B_1,B_2,B_3,B_4\}$ and the 9^{th} point of the STS(9) be 0. Then without loss of generality four triples of the STS(9) can be taken to be $\{0,A_1,B_1\}$, $\{0,A_2,B_2\}$, $\{0,A_3,B_3\}$, $\{0,A_4,B_4\}$. Now there are 6 pairs $A_iA_j, 1\le i<j\le 4$ and 6 pairs $B_iB_j, 1\le i<j\le 4$, so

there must be a block $\{A_i, A_j, A_k\}$ or $\{B_i, B_j, B_k\}, i \neq j \neq k \neq i$, in the Steiner triple system which cannot be used to represent the graph.

Representation of the graphs C4 and C7 are given in Figure 7.4 below, using the STS(9) with block set $\{0,1,2\}$, $\{3,4,5\}$, $\{6,7,8\}$, $\{0,3,6\}$, $\{1,4,7\}$, $\{2,5,8\}$, $\{0,4,8\}$, $\{1,5,6\}$, $\{2,3,7\}$, $\{0,5,7\}$, $\{1,3,8\}$, $\{2,4,6\}$. The disconnected graph consisting of two copies of K_4 can be represented by labelling the vertices of one of the K_4s with the points $1,2,3,6$ and the other with the points $4,5,7,8$.

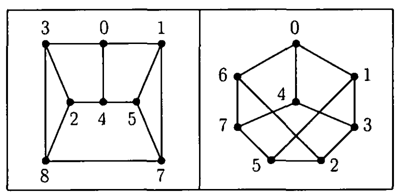


Figure 7.4: Representation of two connected cubic graphs on 8 vertices.

Finally, in this section, it would be remiss to not mention the Petersen graph. This has 10 vertices and 10 edges, so any representation in a Steiner triple system must have at least 10 points and 10 blocks. By modulus constraints, the smallest order of such a Steiner triple system must have 13 points and 26 blocks. There are exactly two Steiner triple systems of this order, one cyclic and the other not. The Petersen graph can be represented in both of these but in fact we can prove more.

Lemma 7.2.5 *Every cubic graph of order* 10 *can be represented in both Steiner triple systems of order* 13.

Proof These are the graphs C9 to C27 given in [51], page 127 as well as the disjoint union of K_4 with either of the two cubic graphs on 6 vertices; 21 graphs in total.

Let the points of the cyclic Steiner triple system be elements of $\mathbb{Z}_{13}$, and let the blocks be the cyclic shifts of $\{0,1,4\}$, $\{0,2,8\}$. The non-cyclic STS(13) can be obtained by choosing any Pasch configuration in the cyclic STS(13) and replacing it with the opposite Pasch configuration. We will choose the blocks $\{2,3,6\}$, $\{2,4,10\}$, $\{3,4,7\}$, $\{6,7,10\}$; replacing them with the blocks $\{2,3,4\}$, $\{2,6,10\}$, $\{3,6,7\}$, $\{4,7,10\}$.

Thus, the cyclic and the non-cyclic STS(13) have 22 blocks in common. Below we give representations of the 21 graphs using just the 22 common blocks. For each graph we list the 15 edges; it is easy to check that they give the required graph and that each edge pair is contained in a different block.

C9: $\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,6\}$, $\{1,9\}$, $\{2,5\}$, $\{2,7\}$, $\{3,5\}$, $\{3,10\}$, $\{5,7\}$, $\{6,9\}$, $\{6,11\}$, $\{7,11\}$, $\{9,10\}$, $\{10,11\}$

C10: $\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,2\}$, $\{1,11\}$, $\{2,12\}$, $\{3,5\}$, $\{3,8\}$, $\{4,8\}$, $\{4,11\}$, $\{4,12\}$, $\{5,6\}$, $\{5,12\}$, $\{6,8\}$, $\{6,11\}$

C11: $\{0,2\}$, $\{0,3\}$, $\{0,7\}$, $\{1,6\}$, $\{1,7\}$, $\{1,9\}$, $\{2,5\}$, $\{2,9\}$, $\{3,5\}$, $\{3,8\}$, $\{4,6\}$, $\{4,8\}$, $\{4,9\}$, $\{5,6\}$, $\{7,8\}$

C12: $\{0,2\}$, $\{0,3\}$, $\{0,6\}$, $\{1,3\}$, $\{1,4\}$, $\{1,6\}$, $\{2,5\}$, $\{2,7\}$, $\{3,5\}$, $\{4,8\}$, $\{4,9\}$, $\{5,7\}$, $\{6,9\}$, $\{7,8\}$, $\{8,9\}$

C13: $\{0,2\}$, $\{0,3\}$, $\{0,4\}$, $\{1,3\}$, $\{1,8\}$, $\{1,11\}$, $\{2,5\}$, $\{2,7\}$, $\{3,5\}$, $\{4,5\}$, $\{4,11\}$, $\{7,8\}$, $\{7,12\}$, $\{8,12\}$, $\{11,12\}$

C14: $\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,2\}$, $\{1,9\}$, $\{2,9\}$, $\{3,5\}$, $\{3,8\}$, $\{4,8\}$, $\{4,9\}$, $\{4,12\}$, $\{5,7\}$, $\{5,12\}$, $\{7,8\}$, $\{7,12\}$

C15: $\{0,2\}$, $\{0,3\}$, $\{0,4\}$, $\{1,2\}$, $\{1,11\}$, $\{1,12\}$, $\{2,12\}$, $\{3,9\}$, $\{3,10\}$, $\{4,5\}$, $\{4,11\}$, $\{5,10\}$, $\{5,11\}$, $\{9,10\}$, $\{9,12\}$

C16: $\{0,1\}$, $\{0,2\}$, $\{0,10\}$, $\{1,3\}$, $\{1,7\}$, $\{2,7\}$, $\{2,11\}$, $\{3,10\}$, $\{3,12\}$, $\{5,8\}$, $\{5,11\}$, $\{5,12\}$, $\{7,8\}$, $\{8,12\}$, $\{10,11\}$

C17: $\{0,1\}$, $\{0,2\}$, $\{0,12\}$, $\{1,2\}$, $\{1,9\}$, $\{2,9\}$, $\{4,5\}$, $\{4,6\}$, $\{4,9\}$, $\{5,6\}$, $\{5,7\}$, $\{6,11\}$, $\{7,11\}$, $\{7,12\}$, $\{11,12\}$

C18: $\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,6\}$, $\{1,11\}$, $\{2,5\}$, $\{2,12\}$, $\{3,5\}$, $\{3,8\}$, $\{4,8\}$, $\{4,11\}$, $\{4,12\}$, $\{5,6\}$, $\{6,11\}$, $\{8,12\}$

C19:	$\{0,2\}$, $\{0,3\}$, $\{0,4\}$, $\{1,2\}$, $\{1,8\}$, $\{1,9\}$, $\{2,7\}$, $\{3,5\}$, $\{3,8\}$, $\{4,9\}$, $\{4,12\}$, $\{5,9\}$, $\{5,12\}$, $\{7,8\}$, $\{7,12\}$
C20:	$\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,2\}$, $\{1,9\}$, $\{2,7\}$, $\{3,5\}$, $\{3,8\}$, $\{5,7\}$, $\{5,9\}$, $\{7,12\}$, $\{8,11\}$, $\{8,12\}$, $\{9,11\}$, $\{11,12\}$
C21:	$\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,2\}$, $\{1,9\}$, $\{2,9\}$, $\{3,5\}$, $\{3,10\}$ $\{5,7\}$, $\{5,12\}$, $\{7,11\}$, $\{7,12\}$, $\{9,10\}$, $\{10,11\}$, $\{11,12\}$
C22:	$\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,2\}$, $\{1,9\}$, $\{2,9\}$, $\{3,5\}$, $\{3,8\}$ $\{5,6\}$, $\{5,12\}$, $\{6,8\}$, $\{6,11\}$, $\{8,12\}$, $\{9,11\}$, $\{11,12\}$
C23:	$\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,5\}$, $\{1,8\}$, $\{2,9\}$, $\{2,11\}$, $\{3,5\}$, $\{3,9\}$, $\{5,12\}$, $\{6,9\}$, $\{6,11\}$, $\{6,12\}$, $\{8,11\}$, $\{8,12\}$
C24:	$\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,5\}$, $\{1,12\}$, $\{2,9\}$, $\{2,11\}$, $\{3,5\}$, $\{3,9\}$, $\{5,8\}$, $\{6,9\}$, $\{6,11\}$, $\{6,12\}$, $\{8,11\}$, $\{8,12\}$
C25:	$\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{1,2\}$, $\{1,11\}$, $\{2,11\}$, $\{3,9\}$, $\{3,11\}$, $\{4,8\}$, $\{4,9\}$, $\{4,12\}$, $\{8,10\}$, $\{8,12\}$, $\{9,10\}$, $\{10,12\}$
C26:	$\{0,2\}$, $\{0,3\}$, $\{0,4\}$, $\{2,9\}$, $\{2,12\}$, $\{3,5\}$, $\{3,9\}$, $\{4,8\}$, $\{4,9\}$, $\{5,6\}$, $\{5,7\}$, $\{6,8\}$, $\{6,12\}$, $\{7,8\}$, $\{7,12\}$
C27:	$\{0,2\}$, $\{0,3\}$, $\{0,4\}$, $\{2,7\}$, $\{2,11\}$, $\{3,8\}$, $\{3,9\}$, $\{4,5\}$, $\{4,6\}$, $\{5,7\}$, $\{5,9\}$, $\{6,8\}$, $\{6,11\}$, $\{7,8\}$, $\{9,11\}$
$K_4 \cup C2$:	$\{0,1\}$, $\{0,2\}$, $\{0,7\}$, $\{1,2\}$, $\{1,7\}$, $\{2,7\}$, $\{3,8\}$, $\{3,9\}$, $\{3,12\}$, $\{5,8\}$, $\{5,9\}$, $\{5,11\}$, $\{8,12\}$, $\{9,11\}$, $\{11,12\}$
$K_4 \cup C3$:	$\{0,1\}$, $\{0,2\}$, $\{0,7\}$, $\{1,2\}$, $\{1,7\}$, $\{2,7\}$, $\{3,9\}$, $\{3,11\}$, $\{3,12\}$, $\{6,9\}$, $\{6,11\}$, $\{6,12\}$, $\{9,10\}$, $\{10,11\}$, $\{10,12\}$

Note that C27 is the Petersen graph. ■

7.3 Graphs of bounded degree

Representing the complete graph K_n in a Steiner triple system is equivalent to finding an independent set. There are interesting things to say about representing other classes of graphs. We consider graphs of a given maximum degree Δ, beginning by representing cycles.

In a Steiner triple system let $t(x, y)$ denote the third point in the block containing x and y. We first prove an easy result.

Theorem 7.3.1 *Every cycle C_n can be represented in every Steiner triple system of order $m \geq n+3$.*

Proof The proof is by induction on n. To start the induction at $n = 3$, pick any two points x, y and $z \neq t(x, y)$. These three points represent C_3.

For the inductive step, suppose that we have a representation of C_n, say by points $1, 2, \ldots, n$ in cyclic order. Pick a point x that is not equal to any of $\{1, 2, \ldots, n\} \cup t(1, 2) \cup t(n-1, n) \cup t(1, n)$. Such a choice is possible provided $m > n+3$. Now remove the edge $\{1, n\}$ from C_n and add the edges $\{1, x\}$, $\{x, n\}$. Using the blocks with these edges gives a representation of C_{n+1}, in particular since $x \neq t(1, 2)$, the block containing $1, 2$ and the block containing $1, x$ are distinct, the same holds for the blocks containing the pairs x, n and $n, n-1$. Furthermore, the blocks containing $1, x$ and x, n are also distinct. Note that the inductive step breaks down when $m = n+3$. ■

However, we can do much better but there are some exceptions which we give first in the following proposition.

Proposition 7.3.2

1. *C_3 cannot be represented in the* STS(3).
2. *$C_3 \cup C_4$ cannot be represented in the* STS(7).

Proof

1. The STS(3) contains only one block; therefore it is impossible to represent the

three edges of C_3.

2. Suppose that there exists a representation of $C_3 \cup C_4$ in the STS(7). Then each block represents an edge since $|B| = |E|$. Now suppose that C_4 is represented by (x, u, y, v). There exists a block $\{x, y, z\}$ where $z \neq u$ or v but this block cannot represent any of the pairs $\{x, y\}$, $\{x, z\}$ or $\{y, z\}$. Contradiction. ∎

Without loss of generality, let the blocks containing the point 0 be $\{0, 2i-1, 2i\}$, $1 \leq i \leq (m-1)/2$. Let $G = (V, E)$ be a disjoint union of cycles where the total number of vertices is n. The cycles will be of three types:

(i) even cycles $C_{x_1}, C_{x_2}, \ldots, C_{x_p}$

(ii) triangles $T_1, T_2, \ldots, T_q$

(iii) odd cycles $C_{y_1}, C_{y_2}, \ldots, C_{y_r}$ of length ≥ 5.

So $\displaystyle\sum_{i=1}^{p} x_i + \sum_{i=1}^{r} y_i + 3q = n.$

Theorem 7.3.3 *Every disjoint union of cycles G where the total number of vertices is n can be represented in every Steiner triple system of order $m \geq n$ except for $(G, m) = (C_3, 3)$ and $(C_3 \cup C_4, 7)$.*

Proof If $m \geq n+1$ then the following algorithm gives the representation,

input : Disjoint union of cycles G, Steiner triple system of order m
output: A representation of the cycles by the Steiner triple system

$c \leftarrow 0$

for $i \leftarrow 1$ **to** p **do**
  represent C_{x_i} by $(c+1, c+2, \ldots, c+x_i)$
  $c \leftarrow c + x_i$
end

for $i \leftarrow 1$ **to** q **do**
  represent T_i by $(c+1, c+2, c+3)$
  $c \leftarrow c + 3$
end

```
for i ← 1 to r do
    if c is even then
        if {c+1, c+y_i, c+y_i−1} ∉ B then
            represent C_{y_i} by (c+1, c+2, ..., c+y_i)
        else
            represent C_{y_i} by (c+2, c+1, ..., c+y_i)
        end
    end
    if c is odd then
        if {c+1, c+2, c+y_i,} ∉ B then
            represent C_{y_i} by (c+1, c+2, c+3, ..., c+y_i)
        else
            represent C_{y_i} by (c+1, c+3, c+2, ..., c+y_i)
        end
    end
    c ← c+y_i
end
```

To complete the proof, we show that a union of cycles can be represented in a Steiner triple system of order $m = n$. We will consider two cases,

(i) Union of cycles where at least one cycle has length ≥ 5.

Take the longest cycle and remove one of its vertices. Denote that cycle by Γ. Apply the algorithm to get a representation of the graph on $m-1$ vertices. It remains to add the vertex, which is 0, back to Γ.

If Γ is of length 4, the cycle will be represented by $(c+1, c+2, c+3, c+4)$. Replace Γ with the cycle $(c+4, c+2, c+3, 0, c+1)$.

If Γ is of even length $2s$, $s \geq 3$, the cycle will be represented by $(c+1, c+2, c+3, c+4, c+5, \ldots, c+2s)$ for some value of c which is even. Replace Γ by the cycle $(c+4, c+2, c+3, 0, c+1, c+5, \ldots, c+2s)$ which is valid provided $\{c+2s, c+4, c+2\}$ is not a block. If it is a block, swap $c+2s$ and $c+2s-1$.

If Γ is of odd length $2s-1$, $s \geq 3$, the cycle will be represented by $(\alpha,\ \beta,\ c+3,\ \ldots,\ c+2s-2,\ c+2s-1)$, $\{\alpha, \beta\} = \{c+1, c+2\}$ for some value of c which is even or represented by $(c+1, \alpha, \beta, c+4, \ldots, c+2s-2, c+2s-1)$, $\{\alpha, \beta\} = \{c+2, c+3\}$ for some value of c which is odd. In the former case, replace Γ by the cycle $(\alpha, \beta, c+3, \ldots, 0, c+2s-1, c+2s-2)$ and in the latter by the cycle $(c+2s-1, \alpha, \beta, c+4, \ldots, c+2s-2, 0, c+1)$.

(ii) Union of q triangles and p squares.

(a) $q = 1$, $p \geq 2$

Working modulo m apply the above algorithm. The triangle will be represented by $(m-2, m-1, 0)$ which is a block and hence the representation is not valid. Pick 1 and 4 from the square $(1, 2, 3, 4)$ and swap them with $m-1$ and $m-2$. If $\{2, 3, m-1\}$ and $\{2, 3, m-2\} \notin B$ then we have a valid representation. Otherwise we make a further modification. Pick 7 and 8 from the second square $(5, 6, 7, 8)$ and swap them with $m-1$ and $m-2$ in the first square.

(b) $q \geq 2$, $p \geq 0$

Working modulo m apply the above algorithm. The last triangle will be represented by $(m-2, m-1, 0)$ which is a block and hence the representation is not valid. Pick a point x from a different triangle (x, y, z) such that $t(x, y) = 0$ or $t(x, z) = 0$ and swap it with 0. ∎

We next turn our attention to representing a graph of maximum degree Δ.

Theorem 7.3.4 *Let G be any graph of order n and of maximum degree Δ. Then G can be represented in any Steiner triple system of order $m \geq n + (3/2)\Delta(\Delta-1)$.*

Proof We find points, one at a time, to represent vertices in a technique reminiscent of Theorem 7.3.1. Let v be as yet not represented. There are at most $\Delta(\Delta-1)$ edges not incident with v but incident with vertices adjacent to v. Call these edges $\{a_i, b_i\}$. Select a point x not currently representing any of the at most $n-1$ other represented vertices, and not equal to any of the $t(\phi(a_i), \phi(b_i))$ (provided that both vertices have already been represented). In addition, for each of the possible $deg(v)(deg(v)-1)/2$ pairs of edges $\{v, u_i\}, \{v, u_j\}$ incident with v, then $t(x, \phi(u_i)) \neq \phi(u_j)$, i.e. $x \neq t(\phi(u_i), \phi(u_j))$ (again provided that the vertices u_i, u_j have been represented).

This can be done provided $m > n - 1 + \Delta(\Delta - 1) + \Delta(\Delta - 1)/2$. Represent v by an available point x and the edges incident with v by the appropriate blocks (provided that a neighbour of v has already been represented). The result is a one-to-one function on the vertices represented, and the blocks representing edges are disjoint by our restriction on $t(\phi(a_i), \phi(b_i))$ and $t(\phi(u_i), \phi(u_j))$. Continuing in this manner, we finish with a representation of G. ■

Corollary 7.3.5 *If G is 2-regular of order n, then G is represented in any Steiner triple system of order $m \geq n + 3$.*

Note that this result is more general than Theorem 7.3.1 in that G need not be a single cycle. It also has a different proof. But Theorem 7.3.3 is a much stronger result. However, for $\Delta = 3$, we get the following,

Corollary 7.3.6 *If G is cubic of order n, then G is represented in any Steiner triple system of order $m \geq n + 9$.*

For small Δ, Theorem 7.3.4 is stronger than Proposition 7.1.4. For large Δ, Proposition 7.1.4 is stronger. The change over occurs at $\Delta \approx \sqrt{1/3n}$. The reason for this is as follows. As mentioned earlier, the value of the function $g(n)$ from Proposition 7.1.4 cannot be expressed using elementary functions. Therefore, as an approximation use $n^2/2$ from Theorem 7.1.6. For large Δ, the value given in Theorem 7.3.4 is approximately $(3/2)\Delta^2$. Comparing these two values gives the required result.

7.4 Complete bipartite graphs

The last section relates to complete bipartite graphs and specifically the maximal complete bipartite graphs that can be represented in a Steiner triple system of some order. A complete bipartite graph $K_{i,j}$ which can be represented in an STS(m) is said to be *maximal* if the complete bipartite graphs $K_{i+1,j}$ and $K_{i,j+1}$ cannot be represented in the same STS(m). We start by proving a rather easy result.

Lemma 7.4.1 *The complete bipartite graphs $K_{1,(m+1)/2}$ and $K_{2,(m-1)/2}$ cannot be represented in any Steiner triple system of order m.*

Proof The complete bipartite graph $K_{1,(m+1)/2}$ cannot be represented in an STS(m) since the valency of the vertex in the left partition is $(m+1)/2$ which is greater than $(m-1)/2$, the replication number of the STS(m).

Assume that $K_{2,(m-1)/2}$ is represented in an STS(m). Let x and y be the representation of the two vertices in the left partition of $K_{2,(m-1)/2}$. The total number of blocks through x and through y is $m-2$ since they have one block in common. But the number of edges in $K_{2,(m-1)/2}$ is $m-1$. Contradiction. ∎

Corollary 7.4.2 *The complete bipartite graph $K_{1,(m-1)/2}$ can be represented in every STS(m) and is maximal.*

It's easy to see that $K_{1,(m-1)/2}$ can be represented in the STS(m); represent the vertex of maximum valency by any point of the system and the edges by the blocks through that point.

Next, we give the maximal representable bipartite graphs for the Steiner triple systems of order 7, 9, 13, and 15.

Proposition 7.4.3 *The maximal complete bipartite graphs that can be represented in the STS(7) are $K_{1,3}$ and $K_{2,2}$.*

Proof In order for $K_{2,2}$ to be maximal, we must show that $K_{2,3}$ cannot be represented in the STS(7). This follows from the second part of Lemma 7.4.1. The graphs are illustrated below, using the Steiner triple system on elements of $\mathbb{Z}_7$ obtained by cyclic shifts of $\{0, 1, 3\}$. ■

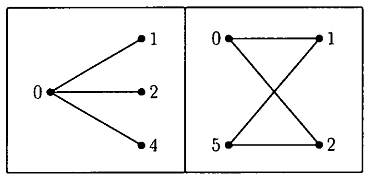


Figure 7.5: The two maximal complete bipartite graphs in the STS(7).

Proposition 7.4.4 *The maximal complete bipartite graphs that can be represented in the $STS(9)$ are $K_{1,4}$ and $K_{3,3}$.*

Proof In order for $K_{3,3}$ to be maximal, we must show that $K_{3,4}$ cannot be represented. Assume the converse. Since $K_{3,4}$ has 12 edges, each block represents an edge. Consider any two vertices x, y from the same partition. The block $\{x, y, z\}$ containing the two vertices cannot represent an edge since if it represents the edge (x, z) then the edge (y, z) is impossible to be represented and vice versa. Contradiction.

The graphs are illustrated below, using the Steiner triple system with block set $\{0, 1, 2\}$, $\{3, 4, 5\}$, $\{6, 7, 8\}$, $\{0, 3, 6\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$, $\{0, 4, 8\}$, $\{1, 5, 6\}$, $\{2, 3, 7\}$, $\{0, 5, 7\}$, $\{1, 3, 8\}$, $\{2, 4, 6\}$. ■

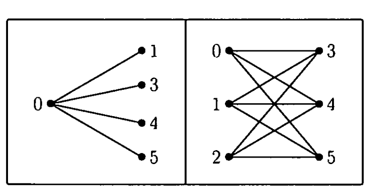


Figure 7.6: The two maximal complete bipartite graphs in the STS(9).

Proposition 7.4.5 *The maximal complete bipartite graphs that can be represented in an $STS(13)$ are $K_{1,6}$, $K_{2,5}$ and $K_{3,4}$.*

Proof To prove that $K_{2,5}$ is maximal we must show that $K_{2,6}$ and $K_{3,5}$ cannot be represented in either STS(13). The former follows from the second part of Lemma 7.4.1. Let $X = \{0,1,2\}, Y = \{3,4,5,6,7\}$ and $Z = \{8,9,10,11,12\}$. Assume that $K_{3,5}$ is represented in the STS(13) as follows: the three vertices in the small partition by the elements of X, and the five vertices in the large partition by the elements of Y. Therefore, there are 15 blocks of the form $\{x,y,z\}$, $x \in X, y \in Y, z \in Z$, representing the edges of the graph. Since five blocks through each point 0, 1 and 2 have been used, the block $\{0,1,2\}$ is forced as the sixth. The remaining blocks of the system contain elements from the sets Y and Z only. The minimum number of blocks formed by pairs of elements of Y is six; for example $\{3,4,5\}$, $\{3,6,7\}$, $\{4,6,z_1\}$, $\{4,7,z_2\}$, $\{5,6,z_3\}$, $\{5,7,z_4\}$, $z_i \in Z$. The minimum number of blocks formed by pairs of elements of Z is also six. This brings the total number of blocks to 28. But an STS(13) has only 26 blocks which leads to a contradiction.

Next, we prove that $K_{3,4}$ is maximal. $K_{3,5}$ has already been proven impossible to represent in both STS(13)s above. We must show that $K_{4,4}$ cannot be represented. Let $X = \{0,1,2,3\}$ and $Y = \{4,5,6,7\}$. Represent one partition of $K_{4,4}$ by elements of X and the other by elements of Y. Then the 16 blocks containing the pairs $\{x,y\}$, $x \in X$, $y \in Y$ are represented. Now consider the blocks containing any two elements from the same partition. The blocks containing the elements of X should be one of the following,

(1) $\{0,1,2\}$, $\{0,3,z\}$, $\{1,3,z\}$, $\{2,3,z\}$, $z \notin X, Y$, without loss of generality, or

(2) $\{0,1,z\}$, $\{0,2,z\}$, $\{0,3,z\}$, $\{1,2,z\}$, $\{1,3,z\}$, $\{2,3,z\}$, $z \notin X, Y$.

The replication number of an STS(13) is 6. In both of the above possibilities, at least one element appears more than 6 times in the system. Contradiction.

The graphs are illustrated below, using the 22 blocks the two Steiner triple systems of order 13 have in common. As noted before, the cyclic STS(13) can be obtained by cyclic shifts of $\{0,1,4\}$ and $\{0,2,8\}$ on elements of $\mathbb{Z}_{13}$. The non-cyclic STS(13) can be obtained by choosing any Pasch configuration in the cyclic

STS(13) and replacing it with the opposite Pasch configuration. We will choose the blocks $\{2,3,6\}$, $\{2,4,10\}$, $\{3,4,7\}$, $\{6,7,10\}$; replacing them with the blocks $\{2,3,4\}$, $\{2,6,10\}$, $\{3,6,7\}$, $\{4,7,10\}$. ■

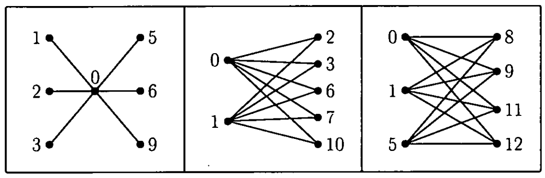


Figure 7.7: The three maximal complete bipartite graphs in the STS(13).

The two Steiner triple systems of order 13 can represent the same maximal complete bipartite graphs. However, this is not true for the Steiner triple systems of order 15. There are 80 nonisomorphic Steiner triple systems of order 15. The graphs $K_{1,7}$, $K_{2,6}$, $K_{3,5}$, and $K_{4,4}$ can be represented in all 80 systems. However, more than half, 54 to be precise, can also represent $K_{3,6}$. These graphs, i.e. $K_{1,7}$, $K_{2,6}$, $K_{3,5}$, $K_{4,4}$ or $K_{1,7}$, $K_{3,6}$, $K_{4,4}$ are the maximal bipartite graphs. To prove maximality we need to show that $K_{4,5}$ cannot be represented.

Proposition 7.4.6 *The complete bipartite graph $K_{4,5}$ cannot be represented in any $STS(15)$.*

Proof Assume that $K_{4,5}$ is represented in an STS(15) and without loss of generality assume that the four vertices in the small partition are represented by 0, 1, 2 and 3. The blocks containing the pairs $\{x_1, x_2\}$, $x_1, x_2 \in \{0,1,2,3\}, x_1 \neq x_2$, are not used in the representation. These pairs can occur in four or six blocks. If they occur in four blocks then one of the points appears in three of these blocks. Since the replication number of an STS(15) is 7, then there are four more blocks through that point in the system. But this is a contradiction since the valency of the vertex represented by that point is five. If they occur in six blocks then every point appears in three of these blocks. The same argument applies as above. ■

Using the standard listing of the STS(15)s given in [44], Appendix C gives the number of representations of the complete bipartite graphs $K_{1,7}$, $K_{2,6}$, $K_{3,5}$, $K_{3,6}$

and $K_{4,4}$. Below we illustrate the graphs using system #11 which is the first in the list that represents all five.

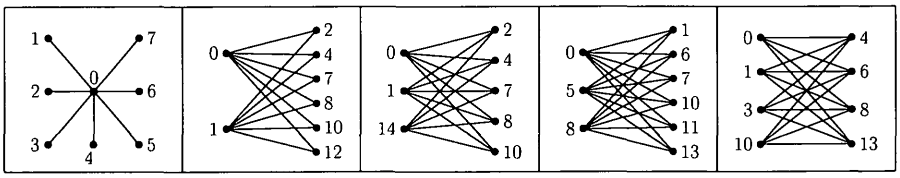


Figure 7.8: Complete bipartite graphs in the STS(15) #11.

CHAPTER 8

Enumerating graph representations

An *ℓ-line configuration* in an STS(m) is any collection of ℓ blocks of the Steiner triple system. For some configurations, the number of occurrences in an STS(m) can be expressed as a rational polynomial in m. Thus, for any admissible m this number is the same regardless of the structure of the STS(m). Such configurations are called *constant* whereas other configurations are called *variable*. Configurations and their occurrences in Steiner triple systems have been studied; the articles [11, 26] are some examples.

In this chapter we extend this study to graphs. In other words, we consider the following question: Given a Steiner triple system of order m and a graph G on $n \leq m$ vertices what is the number of occurrences of G in the STS(m)? This question arose during the study of representing graphs in Steiner triple systems. Consequently, the number of occurrences of a graph in a Steiner triple system is the number of the different representations of that graph by that Steiner triple system. As with configurations, we will refer to a graph as constant or variable. The first section involves the enumeration of graphs with up to three edges.

8.1 One, two and three-edge graphs

There is only one one-edge graph, i.e. a single edge. In any STS(m) there are $\binom{m}{2} = m(m-1)/2$ single edges. A different way of counting the one-edge graphs is by using the one-line configurations; in any STS(m) there are $m(m-1)/6$ one-line configurations and every one-line configuration contains a single edge three times which gives the same result.

There are two two-edge graphs; a pair of disjoint edges denoted by B_1 and a path of length 2 denoted by B_2. The graph B_1 can be obtained by adding an extra edge to the one-edge graph. The number of choices for the extra edge is $\binom{m-2}{2} = (m-2)(m-3)/2$. But the graph B_1 can arise in two ways, hence

$$b_1 = [m(m-1)(m-2)(m-3)/4]/2 = m(m-1)(m-2)(m-3)/8.$$

The graph B_2 can be obtained by adjoining an extra edge through one of the two vertices of the one-edge graph. The number of choices for the extra edge is $4[(m-1)/2-1]$ and again the graph B_2 can arise in two ways, hence

$$b_2 = 4[(m-1)/2-1]m(m-1)/4 = m(m-1)(m-3)/2.$$

Similarly as above, we will give an alternative way of counting the graphs. There are two two-line configurations; a pair of disjoint blocks denoted by B_1' and a pair of intersecting blocks denoted by B_2'. These configurations are constant and the number of occurrences is given by

$$b_1' = m(m-1)(m-3)(m-7)/72, \quad b_2' = m(m-1)(m-3)/8.$$

The graph B_1 occurs in B_1' in nine ways and in B_2' in five ways. Therefore,

$$b_1 = 9b_1' + 5b_2' = m(m-1)(m-2)(m-3)/8.$$

Similarly, the graph B_2 cannot occur in B_1' but occurs in B_2' in four ways. Therefore,

$$b_2 = 4b_2' = m(m-1)(m-3)/2.$$

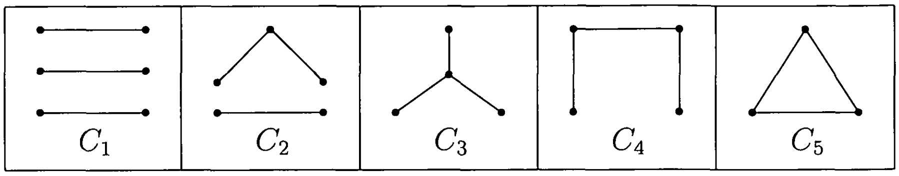


Figure 8.1: The three-edge graphs.

Finally, there are five three-edge graphs; these are given in Figure 8.1 and are denoted by $C_1, C_2, \ldots, C_5$. The graph C_1 can be obtained by adding an extra edge to the graph B_1. The number of choices for the extra edge is $\binom{m-4}{2} = (m-4)(m-5)/2$. The graph C_1 can arise in three ways, so

$$c_1 = [b_1(m-4)(m-5)/2]/3 = m(m-1)(m-2)(m-3)(m-4)(m-5)/48.$$

The graph C_2 can be obtained by adding an extra edge to the graph B_2. The number of choices for the extra edge is $\binom{m-3}{2} = (m-3)(m-4)/2$ and so

$$c_2 = b_2(m-3)(m-4)/2 = m(m-1)(m-3)^2(m-4)/4.$$

The graph C_3 can be obtained by adjoining an extra edge to the common vertex of graph B_2. There are $2[(m-1)/2-2] = m-5$ different ways to adjoin the extra edge and the graph can arise in three ways, so

$$c_3 = b_2(m-5)/3 = m(m-1)(m-3)(m-5)/6.$$

To obtain the graph C_4 we start with the graph B_2 and add an extra edge to one of its vertices of valency one. This can be done in $2[2[(m-1)/2-1]-1] = 2(m-4)$ ways. The graph can arise in two ways, therefore,

$$c_4 = 2b_2(m-4)/2 = m(m-1)(m-3)(m-4)/2.$$

Finally, consider the graph C_5. To obtain it, start again with the graph B_2 and add the edge joining the two vertices of valency one. But the graph can arise in three ways. Hence,

$$c_5 = b_2/3 = m(m-1)(m-3)/6.$$

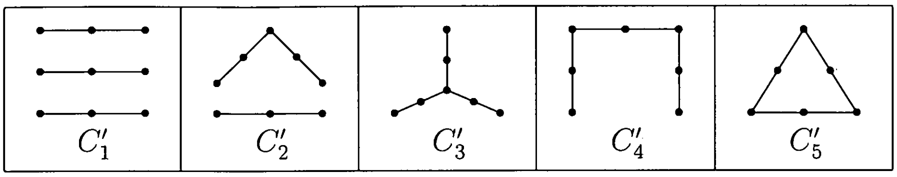


Figure 8.2: The three-line configurations.

We will now check the above results using configurations. There are five three-line configurations; these are given in Figure 8.2 and are denoted by $C'_1, C'_2, \ldots, C'_5$.

The number of occurrences of each five-line configuration in an STS(m) is,

$$c'_1 = m(m-1)(m-3)(m-7)(m^2-19m+96)/1296$$
$$c'_2 = m(m-1)(m-3)(m-7)(m-9)/48$$
$$c'_3 = m(m-1)(m-3)(m-5)/48$$
$$c'_4 = m(m-1)(m-3)(m-7)/8$$
$$c'_5 = m(m-1)(m-3)/6$$

In the table below we list the number of occurrences of every graph in each of the five configurations.

	C_1	C_2	C_3	C_4	C_5
C'_1	27	·	·	·	·
C'_2	15	12	·	·	·
C'_3	7	12	8	·	·
C'_4	7	16	·	4	·
C'_5	2	15	·	9	1

Hence, the number of occurrences of each five-edge configuration in an STS(m) is,

$$c_1 = 27c'_1 + 15c'_2 + 7c'_3 + 7c'_4 + 2c'_5 = m(m-1)(m-2)(m-3)(m-4)(m-5)/48$$
$$c_2 = 12c'_2 + 12c'_3 + 16c'_4 + 15c'_5 = m(m-1)(m-3)^2(m-4)/4$$
$$c_3 = 8c'_3 = m(m-1)(m-3)(m-5)/6$$
$$c_4 = 4c'_4 + 9c'_5 = m(m-1)(m-3)(m-4)/2$$
$$c_5 = c'_5 = m(m-1)(m-3)/6.$$

We have now shown that the number of occurrences of every n-edge graph, when $n \leq 3$, in a Steiner triple system of any order m is constant. This is not a surprise since the number of occurrences of every ℓ-line configuration, when $\ell \leq 3$, in a Steiner triple system of any order m is constant as well. However, we know from [26] that not all four-line configurations are constant. Indeed this is also the case for five-line and six-line configurations. In the next sections we investigate if this is also true for graphs on 4, 5 and 6 edges.

8.2 Four-edge graphs

There are 16 four-line configurations. These are shown in Figure 8.3 and are denoted by $D'_1, D'_2, \dots D'_{16}$. We know from [26] that five of them are constant and all the others are variable. The constant four-line configurations are D'_4, D'_7, D'_8, D'_{11}, and D'_{15}.

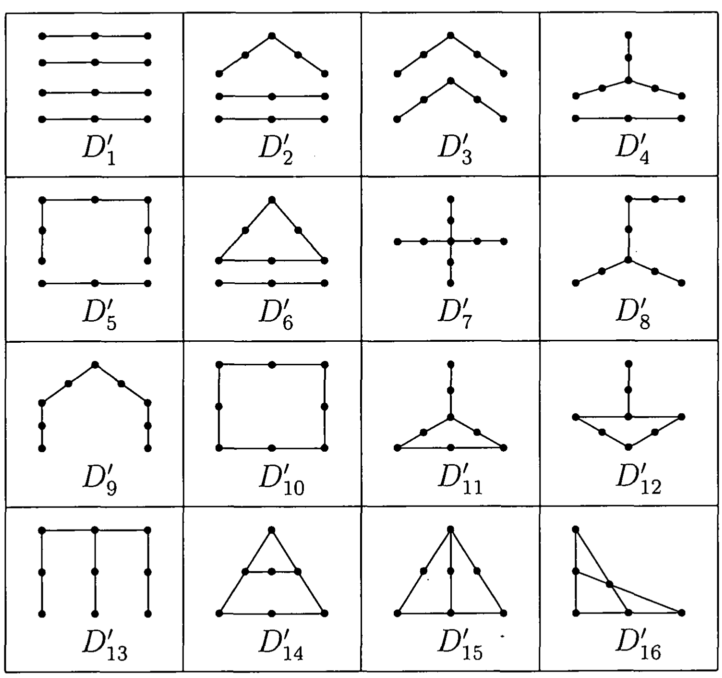


Figure 8.3: The four-line configurations.

Note that D'_{16} is the Pasch configuration denoted by p. The formulae for the numbers of four-line configurations in an STS(m) are given below.

$$d'_1 = m(m-1)(m-3)(m-9)(m-10)(m-13)(m^2-22m+141)/31104 + p$$

$$d'_2 = m(m-1)(m-3)(m-9)(m-10)(m^2-22m+129)/576 - 6p$$

$$d'_3 = m(m-1)(m-3)(m-9)^2(m-11)/128 + 3p$$

$$d'_4 = m(m-1)(m-3)(m-7)(m-9)(m-11)/288$$

$$d'_5 = m(m-1)(m-3)(m-9)(m^2-20m+103)/48 + 12p$$

$$d'_6 = m(m-1)(m-3)(m-9)(m-10)/36 - 4p$$

$$d'_7 = m(m-1)(m-3)(m-5)(m-7)/384$$

$$d'_8 = m(m-1)(m-3)(m-7)(m-9)/16$$

$$d'_9 = m(m-1)(m-3)(m-9)^2/8 - 12p$$

$$d'_{10} = m(m-1)(m-3)(m-8)/8 + 3p$$

$$d'_{11} = m(m-1)(m-3)(m-7)/4$$

$$d'_{12} = m(m-1)(m-3)(m-9)/4 + 12p$$

$$d'_{13} = m(m-1)(m-3)(m^2-18m+85)/48 - 4p$$

$$d'_{14} = m(m-1)(m-3)/4 - 6p$$

$$d'_{15} = m(m-1)(m-3)/6$$

$$d'_{16} = p$$

The number of occurrences of the Pasch configuration in an STS(m), together with the order m, determines the number of occurrences of all the other variable configurations. It will be interesting to see if this is true for the four-edge graphs. There are 11 four-edge graphs and these are shown in Figure 8.4 and are denoted by D_1, D_2, …, D_{11}.

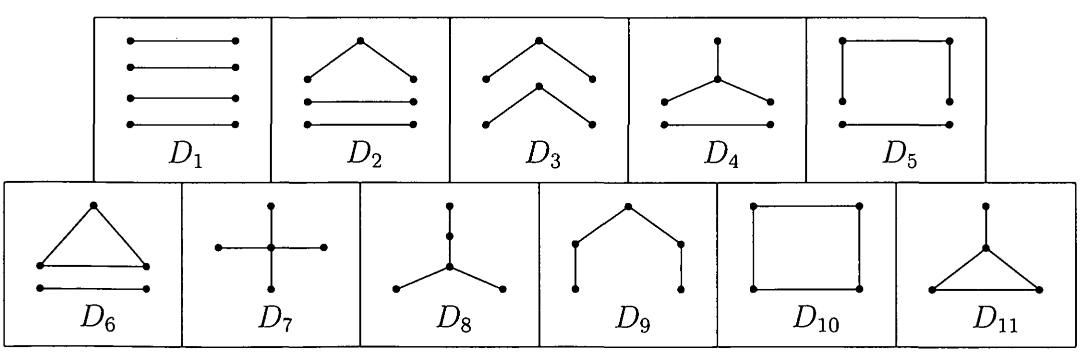


Figure 8.4: The four-edge graphs.

The table below gives the number of occurrences of every four-edge graph in each of the four-line configurations. These results were obtained by hand and were later checked computationally.

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}
D'_1	81	·	·	·	·	·	·	·	·	·	·
D'_2	45	36	·	·	·	·	·	·	·	·	·
D'_3	25	40	16	·	·	·	·	·	·	·	·
D'_4	21	36	·	24	·	·	·	·	·	·	·
D'_5	21	48	·	·	12	·	·	·	·	·	·
D'_6	6	45	·	·	27	3	·	·	·	·	·
D'_7	9	24	·	32	·	·	16	·	·	·	·
D'_8	9	32	8	16	8	·	·	8	·	·	·
D'_9	9	40	12	·	16	·	·	·	4	·	·
D'_{10}	2	28	18	·	20	·	·	·	12	1	·
D'_{11}	2	23	12	10	15	1	·	12	4	·	2
D'_{12}	2	25	8	·	35	3	·	·	8	·	·
D'_{13}	9	36	·	·	36	·	·	·	·	·	·
D'_{14}	·	10	10	·	34	6	·	·	20	1	·
D'_{15}	·	9	12	6	21	3	·	12	12	·	6
D'_{16}	·	·	6	·	24	12	·	·	36	3	·

Using the formulae for the four-line configurations we can easily obtain the formulae for the numbers of four-edge graphs in an STS(m).

$d_1 = m(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)(m-7)/384$

$d_2 = m(m-1)(m-3)^2(m-4)(m-5)(m-6)/16$

$d_3 = m(m-1)(m-3)(m^3-13m^2+57m-87)/8$

$d_4 = m(m-1)(m-3)(m-4)(m-5)^2/12$

$d_5 = m(m-1)(m-3)(m-4)^2(m-5)/4$

$d_6 = m(m-1)(m-3)^2(m-4)/12$

$d_7 = m(m-1)(m-3)(m-5)(m-7)/24$

$d_8 = m(m-1)(m-3)(m-5)^2/2$

$d_9 = m(m-1)(m-3)(m^2-9m+21)/2$

$d_{10} = m(m-1)(m-3)(m-6)/8$

$d_{11} = m(m-1)(m-3)(m-5)/2$

The results show that the number of any four-edge graph in an STS(m) is constant and thus independent of the number of occurrences of the Pasch configuration in the Steiner triple system.

8.3 Five and six-edge graphs

We now consider five-edge graphs. There are 26 five-edge graphs denoted by E_1, E_2, ... E_{26}. For each of these five-edge graphs we list the edges in Table 8.1. They are listed by ascending order of the number of vertices in each graph.

E_1 : 01 02 03 13 23	E_2 : 01 04 12 23 34	E_3 : 01 02 03 24 34
E_4 : 01 02 03 14 23	E_5 : 01 02 03 04 34	E_6 : 01 02 03 23 34
E_7 : 01 02 12 34 35	E_8 : 01 02 13 23 45	E_9 : 01 02 03 23 45
E_{10} : 01 02 03 04 05	E_{11} : 01 02 03 04 15	E_{12} : 01 02 03 14 15
E_{13} : 01 02 13 14 45	E_{14} : 01 02 03 14 45	E_{15} : 01 12 23 34 45
E_{16} : 01 02 03 45 46	E_{17} : 01 02 13 45 46	E_{18} : 01 02 12 34 56
E_{19} : 01 02 03 04 56	E_{20} : 01 02 13 24 56	E_{21} : 01 02 03 14 56
E_{22} : 01 02 34 35 67	E_{23} : 01 02 03 45 67	E_{24} : 01 02 13 45 67
E_{25} : 01 02 34 56 78	E_{26} : 01 23 45 67 89	

Table 8.1: The five-edge graphs.

Similarly, in Table 8.2 we list the blocks of each of the 56 five-line configurations denoted by E'_1, E'_2, ... E'_{56}. These are ordered, as in [11], by ascending order of the number of points in each.

E'_1 : 012 034 135 236 456 E'_2 : 012 034 135 236 146 E'_3 : 012 034 135 236 147

E'_4 : 012 034 135 236 457	E'_5 : 012 034 135 245 067	E'_6 : 012 034 135 246 257
E'_7 : 012 034 135 246 567	E'_8 : 012 034 135 067 168	E'_9 : 012 034 135 067 268
E'_{10} : 012 034 135 067 568	E'_{11} : 012 034 135 236 078	E'_{12} : 012 034 135 236 378
E'_{13} : 012 034 135 236 478	E'_{14} : 012 034 135 245 678	E'_{15} : 012 034 135 246 078
E'_{16} : 012 034 135 246 178	E'_{17} : 012 034 135 246 578	E'_{18} : 012 034 135 267 468
E'_{19} : 012 034 156 357 468	E'_{20} : 012 034 056 178 379	E'_{21} : 012 034 135 067 089
E'_{22} : 012 034 135 067 189	E'_{23} : 012 034 135 067 289	E'_{24} : 012 034 135 067 589
E'_{25} : 012 034 135 067 689	E'_{26} : 012 034 135 236 789	E'_{27} : 012 034 135 246 789
E'_{28} : 012 034 135 267 289	E'_{29} : 012 034 135 267 489	E'_{30} : 012 034 135 267 689
E'_{31} : 012 034 156 357 289	E'_{32} : 012 034 156 378 579	E'_{33} : 012 034 056 078 09a
E'_{34} : 012 034 056 078 19a	E'_{35} : 012 034 056 178 19a	E'_{36} : 012 034 056 178 29a
E'_{37} : 012 034 056 178 39a	E'_{38} : 012 034 056 178 79a	E'_{39} : 012 034 135 067 89a
E'_{40} : 012 034 135 267 89a	E'_{41} : 012 034 135 678 69a	E'_{42} : 012 034 156 278 39a
E'_{43} : 012 034 156 357 89a	E'_{44} : 012 034 156 378 59a	E'_{45} : 012 034 056 078 9ab
E'_{46} : 012 034 056 178 9ab	E'_{47} : 012 034 056 789 7ab	E'_{48} : 012 034 135 678 9ab
E'_{49} : 012 034 156 278 9ab	E'_{50} : 012 034 156 378 9ab	E'_{51} : 012 034 156 789 7ab
E'_{52} : 012 034 056 789 abc	E'_{53} : 012 034 156 789 abc	E'_{54} : 012 034 567 589 abc
E'_{55} : 012 034 567 89a bcd	E'_{56} : 012 345 678 9ab cde	

Table 8.2: The five-line configurations.

From the above configurations only five are constant; the formulae are given in [11]. Note that E'_1 is the Mitre configuration and E'_2 is the Mia configuration. The number of Mitre and Pasch configurations, together with the order m, determine the number of a variable five-line configuration in an STS(m). The Tables 8.3 and 8.4 below give the number of occurrences of every five-edge graph in each of the five-line configurations together with the coefficients of v^2p, vp, the Mitre (μ) and the Pasch (p) configuration taken from the formulae.

Using the formulae for the five-line configurations, computational results for the number of occurrences of a five-edge graph have shown that the coefficients of the Mitre and Pasch configuration in the graph formulae sum to zero. Hence, the number of any five-edge graph in an STS(m) is constant.

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}	E_{13}	v^2p	vp	p	μ
E'_1	·	6	·	12	·	·	12	9	24	·	·	·	24	·	·	·	1
E'_2	2	4	4	16	·	12	12	3	24	·	·	2	24	·	·	3	·
E'_3	1	1	2	8	·	10	6	·	16	·	·	5	20	·	·	−12	·
E'_4	·	2	·	6	·	·	6	3	18	·	·	·	20	·	·	−6	−12
E'_5	·	·	4	·	·	·	8	5	12	·	·	·	12	·	3	−21	·
E'_6	·	1	6	4	·	·	8	3	8	·	·	·	10	·	·	−12	·
E'_7	·	2	·	·	·	·	8	9	·	·	·	·	·	·	·	−6	−3
E'_8	1	·	2	4	·	6	2	·	4	·	·	9	20	·	·	6	·
E'_9	·	1	·	4	·	·	4	·	8	·	·	·	16	·	·	36	6
E'_{10}	·	·	4	2	·	·	2	2	4	·	·	·	8	·	·	24	·
E'_{11}	·	·	·	2	·	8	2	·	12	·	·	8	12	·	·	12	·
E'_{12}	·	·	·	·	12	·	·	·	12	·	24	·	·	·	·	·	·
E'_{13}	·	·	·	2	·	·	2	·	16	·	·	·	12	·	·	24	6
E'_{14}	·	·	·	·	·	·	·	9	·	·	·	·	·	1/6	−19/6	14	·
E'_{15}	·	·	·	·	·	·	·	3	12	·	·	·	16	·	−3	27	3
E'_{16}	·	·	2	·	·	·	4	1	6	·	·	·	6	·	−12	108	·
E'_{17}	·	·	·	·	·	·	4	3	·	·	·	·	·	·	−6	66	6
E'_{18}	·	1	·	·	·	·	4	3	·	·	·	·	·	·	·	48	12
E'_{19}	·	·	·	·	·	·	·	9	·	·	·	·	·	·	·	6	2
E'_{20}	·	·	·	·	4	·	·	·	4	·	24	·	·	·	·	·	·
E'_{21}	·	·	·	·	·	4	·	·	4	·	·	12	8	·	·	−12	·
E'_{22}	·	·	·	·	·	·	·	·	6	·	·	·	8	·	12	−156	−12
E'_{23}	·	·	·	·	·	·	·	·	6	·	·	·	·	·	6	−66	·
E'_{24}	·	·	·	2	·	·	2	·	4	·	·	·	12	·	·	−60	−6
E'_{25}	·	·	·	·	·	·	·	·	18	·	·	·	·	·	·	−12	−2
E'_{26}	·	·	·	·	·	·	·	3	·	·	·	·	·	−1	22	−123	−3
E'_{27}	·	·	·	·	·	·	4	·	·	·	·	·	·	·	6	−9	·
E'_{28}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	12	−144	−12
E'_{29}	·	·	·	·	·	·	4	·	·	·	·	·	·	·	12	−192	−18
E'_{30}	·	·	2	·	·	·	·	1	·	·	·	·	6	·	6	−78	·
E'_{31}	·	·	·	·	·	·	·	3	·	·	·	·	·	·	6	−138	−18
E'_{32}	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	−36	−6
E'_{33}	·	·	·	·	·	·	·	·	·	32	·	·	·	·	·	·	·
E'_{34}	·	·	·	·	·	·	·	·	·	·	16	·	·	·	·	·	·
E'_{35}	·	·	·	·	·	·	·	·	·	·	·	16	·	·	·	3	·
E'_{36}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−6	66	3
E'_{37}	·	·	·	·	·	·	·	·	·	·	·	·	8	·	−6	114	6
E'_{38}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−12	132	·
E'_{39}	·	·	·	·	·	·	·	·	6	·	·	·	·	·	−6	102	6
E'_{40}	·	·	·	·	·	·	·	·	·	·	·	·	·	2	−56	414	18
E'_{41}	·	·	·	·	·	·	4	·	·	·	·	·	·	·	−6	90	6
E'_{42}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−24	324	24
E'_{43}	·	·	·	·	·	·	·	3	·	·	·	·	·	1/2	−25/2	108	6
E'_{44}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−24	432	36
E'_{45}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
E'_{46}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	12	−168	−6
E'_{47}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	3	−33	·
E'_{48}	·	·	·	·	·	·	·	·	·	·	·	·	·	−2/3	56/3	−146	−6
E'_{49}	·	·	·	·	·	·	·	·	·	·	·	·	·	−2/3	74/3	−212	−10
E'_{50}	·	·	·	·	·	·	·	·	·	·	·	·	·	−2	80	−810	−42
E'_{51}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	30	−444	−27
E'_{52}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−3	39	1
E'_{53}	·	·	·	·	·	·	·	·	·	·	·	·	·	2	−74	690	30
E'_{54}	·	·	·	·	·	·	·	·	·	·	·	·	·	1/2	−67/2	384	18
E'_{55}	·	·	·	·	·	·	·	·	·	·	·	·	·	−1	40	−381	−15
E'_{56}	·	·	·	·	·	·	·	·	·	·	·	·	·	1/6	−37/6	56	2

Table 8.3: Number of occurrences of graphs E_1 to E_{13} in the five-line configurations.

	E_{14}	E_{15}	E_{16}	E_{17}	E_{18}	E_{19}	E_{20}	E_{21}	E_{22}	E_{23}	E_{24}	E_{25}	E_{26}	v^2p	vp	p	μ
E'_1	·	66	·	30	6	·	42	12	·	·	·	·	·	·	·	·	1
E'_2	24	40	4	24	4	·	28	16	·	·	·	·	·	·	·	3	·
E'_3	20	17	10	35	3	·	27	32	14	4	12	·	·	·	·	−12	·
E'_4	·	34	·	42	6	·	46	26	14	2	18	·	·	·	·	−6	−12
E'_5	24	24	4	24	16	·	64	16	6	·	24	·	·	·	3	−21	·
E'_6	20	27	6	41	7	·	51	16	15	2	18	·	·	·	·	−12	·
E'_7	·	46	·	50	10	·	78	·	16	·	24	·	·	·	·	−6	−3
E'_8	12	6	18	34	·	·	10	36	43	16	10	10	·	·	·	6	·
E'_9	·	15	·	45	2	·	25	36	39	8	29	11	·	·	·	36	6
E'_{10}	16	12	12	40	1	·	32	18	42	8	29	11	·	·	·	24	·
E'_{11}	16	4	16	24	3	·	20	42	26	12	27	9	·	·	·	12	·
E'_{12}	24	·	24	12	3	12	24	24	24	12	27	9	·	·	·	·	·
E'_{13}	·	12	·	32	7	·	44	34	26	8	39	9	·	·	·	24	6
E'_{14}	·	·	·	·	36	·	108	·	18	·	72	·	·	1/6	−19/6	14	·
E'_{15}	·	16	·	32	6	·	36	40	22	4	46	10	·	·	−3	27	3
E'_{16}	20	8	10	28	8	·	42	24	24	4	46	10	·	·	−12	108	·
E'_{17}	·	16	·	36	14	·	76	·	26	·	58	10	·	·	−6	66	6
E'_{18}	·	23	·	49	5	·	57	·	41	·	48	12	·	·	·	48	12
E'_{19}	·	24	·	48	·	·	60	·	42	·	48	12	·	·	·	6	2
E'_{20}	8	·	24	4	1	20	8	32	32	28	21	31	2	·	·	·	·
E'_{21}	8	·	24	12	1	·	8	40	40	28	21	31	2	·	·	−12	·
E'_{22}	·	4	·	24	3	·	20	44	32	14	53	33	2	·	12	−156	−12
E'_{23}	16	·	16	16	3	·	28	20	36	14	53	33	2	·	6	−66	·
E'_{24}	·	4	·	32	1	·	12	34	58	20	25	35	2	·	·	−60	−6
E'_{25}	·	·	·	·	9	·	36	36	36	18	63	27	·	·	·	−12	−2
E'_{26}	·	·	·	·	18	·	60	·	30	·	102	30	·	−1	22	−123	−3
E'_{27}	16	·	16	20	5	·	16	32	28	8	61	35	2	·	6	−9	·
E'_{28}	·	4	·	24	9	·	52	·	32	·	85	35	2	·	12	−144	−12
E'_{29}	·	8	·	36	5	·	32	·	52	·	65	39	2	·	12	−192	−18
E'_{30}	12	4	18	24	·	·	18	20	56	14	28	38	2	·	6	−78	·
E'_{31}	·	8	·	32	·	·	44	·	54	·	60	40	2	·	6	−138	−18
E'_{32}	·	15	·	45	·	·	25	·	75	·	35	45	2	·	·	−36	−6
E'_{33}	·	·	·	·	·	80	·	·	·	80	·	40	11	·	·	·	·
E'_{34}	·	·	16	·	·	32	·	24	24	56	12	52	11	·	·	·	·
E'_{35}	·	·	32	·	·	·	·	32	32	48	16	56	11	·	·	3	·
E'_{36}	·	·	·	16	·	·	·	48	16	24	68	60	11	·	−6	66	3
E'_{37}	·	·	·	16	·	·	4	32	52	32	24	64	11	·	−6	114	6
E'_{38}	8	·	24	8	·	·	8	16	48	24	28	68	11	·	−12	132	·
E'_{39}	·	·	·	·	3	·	12	36	36	30	45	69	6	·	−6	102	6
E'_{40}	·	·	·	·	9	·	24	·	24	·	105	75	6	2	−56	414	18
E'_{41}	·	·	·	36	5	·	·	·	60	·	45	83	10	·	−6	90	6
E'_{42}	·	·	·	16	·	·	24	·	40	·	80	72	11	·	−24	324	24
E'_{43}	·	·	·	·	·	·	36	·	54	·	60	84	6	1/2	−25/2	108	6
E'_{44}	·	4	·	24	·	·	16	·	68	·	40	80	11	·	−24	432	36
E'_{45}	·	·	·	·	·	48	·	·	·	96	·	72	27	·	·	·	·
E'_{46}	·	·	·	·	·	·	·	24	24	48	24	96	27	·	12	−168	−6
E'_{47}	·	·	32	·	·	·	·	·	48	40	·	88	35	·	3	−33	·
E'_{48}	·	·	·	·	9	·	·	·	·	·	81	135	18	−2/3	56/3	−146	−6
E'_{49}	·	·	·	·	·	·	·	·	·	·	108	108	27	−2/3	74/3	−212	−10
E'_{50}	·	·	·	·	·	·	12	·	36	·	48	120	27	−2	80	−810	−42
E'_{51}	·	·	·	16	·	·	·	·	64	·	20	108	35	·	30	−444	−27
E'_{52}	·	·	·	·	·	·	·	·	·	72	·	108	63	·	−3	39	1
E'_{53}	·	·	·	·	·	·	·	·	·	·	36	144	63	2	−74	690	30
E'_{54}	·	·	·	·	·	·	·	·	48	·	·	120	75	1/2	−67/2	384	18
E'_{55}	·	·	·	·	·	·	·	·	·	·	·	108	135	−1	40	−381	−15
E'_{56}	·	·	·	·	·	·	·	·	·	·	·	·	243	1/6	−37/6	56	2

Table 8.4: Number of occurrences of graphs E_{14} to E_{26} in the five-line configurations.

Finally, we consider graphs on six edges. There are 68 six-edge graphs and 282 six-line configurations. Computational results have shown that the number of any six-edge graph is also independent of the STS(m). However, we will not give any details of our calculations, the methods used to obtain the results are as above.

Based on the above results, we state the following theorem.

Theorem 8.1 *Let G be any graph with $|E(G)| \leq 6$ and let T be any Steiner triple system of order m. Then the number of occurrences of G in T is independent of T.*

To summarize, the number of times a graph with up to six edges occurs in a Steiner triple system is constant even though variable configurations with four, five and six lines exist; indeed most four, five and six-line configurations are variable. Naturally, this gives rise to the following question: What is the smallest number of edges of a variable graph? Clearly, it is greater than six but less than or equal to twelve since $K_{2,6}$ is variable (see Appendix C). However, an investigation on the number of occurrences of $K_{2,4}$ in the two STS(13)s and in the 80 STS(15)s shows that in fact it is less or equal to eight. $K_{2,4}$ occurs 1989 times in the cyclic STS(13) and 1974 times is the non-cyclic STS(13); the results for the STS(15)s are given in Appendix D. Hence, the smallest number of edges of a variable graph is seven or eight.

CHAPTER 9

Topological representations

In this chapter, we consider a topological variation of the problem discussed in the last two chapters. Let $G = (V, E)$ be a simple graph and let $T = (P, B)$ be a triangulation of the complete graph K_m in a surface or pseudosurface, where P is the vertex set of K_m and B is the set of triangles. The surface, or indeed the pseudosurface, can be either orientable or nonorientable. Let ϕ be a one-to-one function from $V(G)$ to $P(T)$. This induces a one-to-two relation, which we will also call ϕ, from $E(G)$ to $B(T)$. Now consider the inverse relation $\phi^{-1} : B(T) \to E(G)$. If ϕ^{-1} is a two-to-one function, i.e. two adjacent triangles represent at most one edge, then we say that the triangulation T *represents* the graph G. This is the same concept as before in that no two edges of G are represented by the same triangle.

If the triangulation is face two-colourable, i.e. the triangles of each of the two colour classes form Steiner triple systems (P, B_1) and (P, B_2), then the above definition can be rephrased in the following way: there exists a one-to-one function $\phi : V(G) \to P$ such that both the induced functions $\phi : E(G) \to B_1$ and $\phi : E(G) \to B_2$ are also one-to-one. Throughout this chapter, we denote $t_i(x, y)$, $i = 1, 2$, as the third points in the blocks containing x and y.

A graph may only be represented in a triangulation of K_m if it has at most m

vertices and at most $m(m-1)/6$ edges. As in Chapter 7, we consider maximal representable and minimal non-representable graphs. Additionally, in this chapter, we consider maximum representable graphs since the triangulations impose extra restrictions to representations thus reducing the number of representations compared to Chapter 7. A representable graph in a triangulation of K_m with at least $m-1$ vertices and $m(m-1)/6$ edges is said to be *maximum.* A representable graph is *maximal* if it is not a subgraph of a representable graph and it is not maximum. A representable graph with $m-2$ vertices cannot be maximum since the two points which are not used in the representation and the edge joining them in the triangulation can always be added to the representation. A non-representable graph in a triangulation is said to be *minimal* or an *obstruction* if all of its subgraphs are representable in the triangulation. These graphs can have at most $m+1$ vertices and at most $m(m-1)/6+1$ edges.

This chapter consists of two parts. In the first part, we investigate the maximal and maximum representable and the minimal non-representable graphs in the triangular embeddings of the complete graph K_m, where $m \leq 7$. In the second part, we are concerned with cycles in the triangulations and seek to prove that every cycle of order at most m can be represented in the triangulation of K_m.

9.1 Triangulations of small order

We start with the first two trivial cases, i.e. the triangulations of the complete graphs K_3 and K_4 which both have unique embeddings in the sphere. The figure below shows the embedding of the STS(3) $= \{0, 1, 2\}$ and the MTS(4) $= (0, 1, 2)$, $(0, 3, 1)$, $(0, 2, 3)$, $(1, 3, 2)$.

It is easy to determine the maximum representable graphs and the obstructions in these triangulations and so we state the next two propositions without proof. The given representations of the graphs are based on the embeddings of Figure 9.1.

Figure 9.1: Embeddings of the STS(3) and the MTS(4) in the sphere.

Proposition 9.1.1 *There is exactly one maximum graph that can be represented in the triangular embedding of the complete graph K_3. The embedding has two minimal obstructions and maximal representable graphs do not exist.*

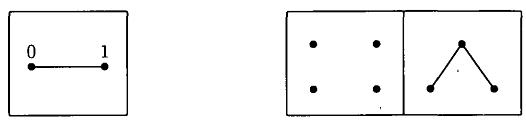


Figure 9.2: Maximum graph and minimal obstructions of the K_3 triangulation.

Proposition 9.1.2 *There is exactly one maximum graph that can be represented in the triangular embedding of the complete graph K_4. The embedding has two minimal obstructions and maximal representable graphs do not exist.*

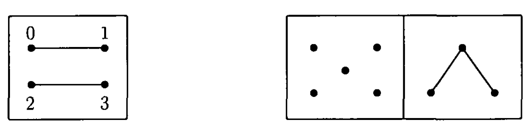


Figure 9.3: Maximum graph and minimal obstructions of the K_4 triangulation.

The next case is the unique twofold triple system of order 6. The TTS(6) = $\{0,1,2\}$, $\{0,1,5\}$, $\{0,2,3\}$, $\{0,3,4\}$, $\{0,4,5\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{2,3,5\}$, $\{2,4,5\}$ has a unique embedding in the projective plane as shown in Figure 9.4. We take a thorough approach to find all minimal obstructions. We start by examining the smallest non-trivial graphs, i.e. with two edges, and continue up to graphs with $m(m-1)/6$ edges. In what follows, the graph references are to the listing in [51].

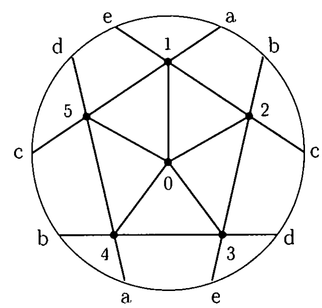


Figure 9.4: Embedding of the TTS(6) in the projective plane.

Proposition 9.1.3 *The triangular embedding of the complete graph K_6 has exactly four minimal obstructions.*

Proof The first minimal obstruction is trivial; it is the graph on 7 vertices and no edges, $\bar{K}_7$, since it has too many vertices. There is just one graph with one edge and two graphs with two edges; all of these can be represented. Similarly, four of the five graphs with three edges can be represented while the fifth cannot. This graph is $K_{1,3}$ and it is non-representable since it has a vertex of valency 3.

Now consider the graphs with four edges. There are two on 4 vertices, G15 and G16 (4-cycle). G15 has $K_{1,3}$ as a subgraph and thus cannot be represented. The 4-cycle is a minimal obstruction and the proof is as follows. Assume that the 4-cycle can be represented by $(0,1,2,3)$ without loss of generality. Then the system contains the blocks $\{0,2,4\}$, $\{0,2,5\}$, $\{1,3,4\}$ and $\{1,3,5\}$ since $t_i(0,2) \neq 1,3$ and $t_i(1,3) \neq 0,2$, $i = 1,2$. This means the pairs $\{0,1\}$, $\{0,3\}$, $\{1,2\}$ and $\{2,3\}$ have to appear twice in distinct blocks. But this would give 12 blocks, contradiction. On 5 vertices, there are four graphs (G29 to G32); G29 and G30 contain $K_{1,3}$ and so are not minimal and the other two, G31 and G32, are representable. Finally, on 6 vertices there are three graphs (G68 to G70); G68 contains $K_{1,3}$ and so is not minimal and G69 is representable. However, G70 is a minimal obstruction and the proof is as follows. G70 consists of two disconnected paths of length 2. Assume they can be represented by $(0,1,2)$ and $(3,4,5)$ so that the vertices of valency 2 are represented by 1 and 4. Then the system has to contain the 8 blocks: $\{0,1,a_i\}$, $\{1,2,b_i\}$, $\{3,4,c_i\}$ and $\{4,5,d_i\}$, where $a_i, b_i \in \{3,4,5\}$ and

$c_i, d_i \in \{0, 1, 2\}$, $i = 1, 2$, and $a_1 \neq a_2$, $b_1 \neq b_2$, $c_1 \neq c_2$, $d_1 \neq d_2$. None of these blocks contain the pairs $\{0, 2\}$ and $\{3, 5\}$ which also have to appear twice in the system in distinct blocks. This brings the number of blocks to 12. Contradiction.

Next consider the graphs with five edges. There are six graphs on 5 or less vertices; five of them (G33 to G37) contain $K_{1,3}$ and the sixth (G38) is representable. All the graphs on 6 vertices contain a minimal obstruction as a subgraph; G77 to G82 contain $K_{1,3}$, G83 and G84 contain G70 and G85 contains the 4-cycle. Finally, consider the graphs with six edges. Each of these graphs contains at least one of the minimal obstructions found above and thus cannot be represented. Hence, the triangular embedding of the complete graph K_6 has four minimal obstructions. These are illustrated below and denoted by A_1 to A_4. ■

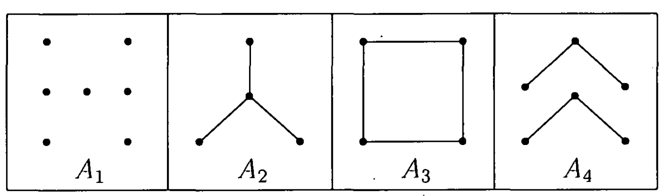


Figure 9.5: The minimal obstructions of the K_6 triangulation.

To find all maximal and maximum graphs we do the opposite of what we did above. More specifically, we start by first examining the graphs on six edges and continue down to graphs with two edges.

Proposition 9.1.4 *There are exactly two maximal graphs and one maximum graph representable in the triangular embedding of the complete graph K_6.*

Proof There are nine graphs on 5 edges and 6 vertices, all of which cannot be represented; six of these have a vertex of valency greater or equal to 3 (G77 to G82), i.e. $K_{1,3}$ is a subgraph of those graphs. Similarly, G83 and G84 contain obstruction A_4 and G85 contains A_3.

There are five graphs on 5 edges and 5 vertices, four of which (G34 to G37) cannot be represented since they contain $K_{1,3}$ as a subgraph but the remaining graph G38, which is the 5-cycle, can be represented and is maximum. The only graph on 5 edges and 4 vertices, G17, contains the minimal obstructions $K_{1,3}$ and A_3.

From the three graphs on 4 edges and 6 vertices only G69 can be represented and is maximal; G68 contains $K_{1,3}$ and G70 is the minimal obstruction A_4. Finally, there are four graphs on 4 edges and 5 vertices. Two of these, G29 and G30, contain $K_{1,3}$ and so cannot be represented whilst G31 is a subgraph of the 5-cycle and so is neither maximum nor maximal. The remaining graph G32 is representable and maximal.

All other graphs of smaller size are either subgraphs of the maximal or maximum graphs or they are non-representable. The maximal and maximum graphs and their representation in the embedding of K_6 are illustrated below. ■

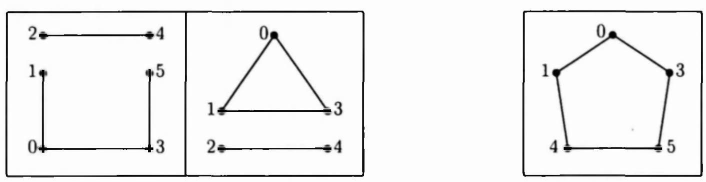


Figure 9.6: The maximal and maximum representable graphs in the K_6 triangulation.

Finally, consider the unique toroidal embedding of the two cyclic Steiner triple systems of order 7 whose blocks are cyclic shifts of the starter blocks $\{0, 1, 3\}$ and $\{0, 2, 3\}$ respectively. We follow the same procedure as above to determine the minimum, maximal and maximum graphs.

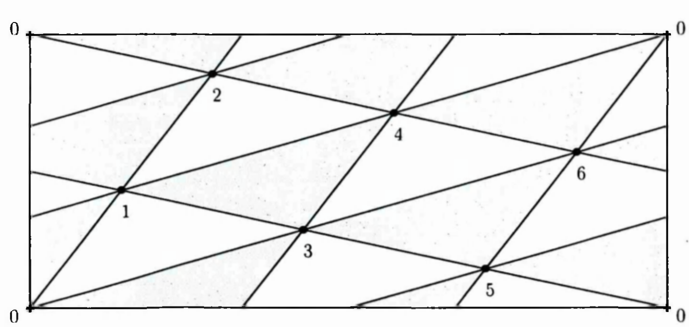


Figure 9.7: The unique toroidal biembedding of the STS(7)s.

Proposition 9.1.5 *The toroidal biembedding of the $STS(7)$s has exactly seven minimal obstructions.*

Proof The first minimal obstruction is the graph on 8 vertices and no edges, $\bar{K}_8$; it is quite clear that this graph cannot be represented because it has too many vertices. Also, it is very easy to see that all the graphs with one, two and three edges are representable.

Now consider the graphs with four edges. There are two on 4 vertices, G15 and G16; the former is representable but the latter, which is the 4-cycle, is non-representable and minimal. The proof is as follows. Suppose the 4-cycle is represented by (a, b, c, d). Then there are blocks $\{a, b, x_1\}$, $\{c, d, x_1\}$, $\{b, c, y_1\}$, $\{d, a, y_1\}$ in the first system and blocks $\{a, b, x_2\}$, $\{c, d, x_2\}$, $\{b, c, y_2\}$, $\{d, a, y_2\}$ in the second system. The first system is uniquely completed by the blocks $\{a, c, z_1\}$, $\{b, d, z_1\}$, $\{x_1, y_1, z_1\}$ and the second system by the blocks $\{a, c, z_2\}$, $\{b, d, z_2\}$, $\{x_2, y_2, z_2\}$. But $\{x_1, y_1, z_1\} = \{x_2, y_2, z_2\}$ since in each system these are the three points different from a, b, c, d. Contradiction. The remaining graphs with four edges are on 5 vertices. There are four of these graphs, G29 to G32. The first is the star $K_{1,4}$ which is non-representable and minimal since it has a vertex of too high a degree. The remaining three are representable.

There are 21 graphs with five edges. The graphs G17, G37 and G85 are non-representable but not minimal since they contain the 4-cycle as a subgraph. Similarly, G34, G77, G78 and G243 are non-representable but not minimal because they contain $K_{1,4}$. Out of the remaining 14 graphs, 12 can be represented and 2 cannot (G79 and G245) and are minimal. The graph G79 is a path of length 3 with pendant edges from the 2nd and 3rd vertices. Suppose it can be represented and suppose that the 2nd and 3rd vertex of the path are represented by x and y respectively and the remaining vertices by a, b, c and d. Therefore, $t_i(x, y) \neq a, b, c, d$, $i = 1, 2$. Consequently, $t_i(x, y) = z$, $i = 1, 2$, where z is the remaining point not represented, a contradiction. The graph G245 is $K_{1,2} \cup K_{1,3}$. Represent the vertex of valency 3 by 0. Then there are two ways to represent $K_{1,3}$, either with $1, 2, 4$ or $3, 5, 6$ as the other vertices. In both cases the remaining three points not represented form a block in one of the systems. Since the systems are cyclic, this will be true for any representation of the vertex of valency 3.

Next, consider the 41 graphs with six edges; 30 of these graphs are non-representable but only two are minimal (G94 and G104). The graph G94 is a triangle with 3 non-adjacent pendant edges. Again, suppose the vertices of the triangle are represented by x, y and z and the vertices at the end of the pendant edges from x, y and z are represented by a, b and c respectively. Let d be the seventh point. Then, $t_i(x,y) \neq a, b, z$, $i = 1, 2$, so the triangulation must contain the triangles $\{x, y, c\}$ and $\{x, y, d\}$. Similarly, it contains the triangles $\{y, z, a\}$, $\{y, z, d\}$, $\{x, z, b\}$ and $\{x, z, d\}$. But now the three blocks containing d cannot be partitioned between two STS(7)s. Contradiction.

The graph G104 is a pentagon with a pendant edge. First consider a path of length 5. Represent it by a, b, c, d, x, y. In order to construct the given graph, the edge $\{a, x\}$ or $\{b, y\}$ has to be added. Let z be the seventh point. Try to add the edge $\{a, x\}$. Now $t_i(x, d) \neq a, c, y$, $i = 1, 2$. The two triangles containing the edge $\{x, d\}$ are thus $\{x, d, z\}$ and $\{x, d, b\}$. Consider the STS(7) containing the block $\{x, d, z\}$. Since $t_i(x, a) \neq b, y$, $i = 1, 2$, the remaining two blocks containing x are $\{x, a, c\}$ and $\{x, b, y\}$. This system, call it the black system, has two completions,

(B1) $\{x, d, z\}$, $\{x, a, c\}$, $\{x, b, y\}$, $\{d, a, b\}$, $\{d, c, y\}$, $\{z, a, y\}$, $\{z, c, b\}$, or

(B2) $\{x, d, z\}$, $\{x, a, c\}$, $\{x, b, y\}$, $\{d, a, y\}$, $\{d, c, b\}$, $\{z, a, b\}$, $\{z, c, y\}$.

But $t_i(b, c) \neq d$, $i = 1, 2$, so only (B1) is a possibility. Now consider the STS(7) containing the blocks $\{x, d, b\}$, the white system. Since $t_i(x, a) \neq y$, $i = 1, 2$ and the block $\{x, a, c\}$ is in (B1) the other two blocks containing x are $\{x, a, z\}$ and $\{x, c, y\}$. Again there are two completions,

(W1) $\{x, d, b\}$, $\{x, a, z\}$, $\{x, c, y\}$, $\{d, a, c\}$, $\{d, z, y\}$, $\{b, a, y\}$, $\{b, z, c\}$, or

(W2) $\{x, d, b\}$, $\{x, a, z\}$, $\{x, c, y\}$, $\{d, a, y\}$, $\{d, z, c\}$, $\{b, a, c\}$, $\{b, z, y\}$.

Again $t_i(b, c) \neq a$, $i = 1, 2$, so only (W1) is a possibility. But both (B1) and (W1) contain the block $\{b, z, c\}$. The same argument applies for the edge $\{b, y\}$.

Finally, consider the graphs on seven and eight edges. The non-representable graphs contain at least one of the minimal obstructions found above and so are not minimal. Therefore, the biembedding of the STS(7)s has seven minimal obstructions. These are illustrated below and denoted by B_1 to B_7. ■

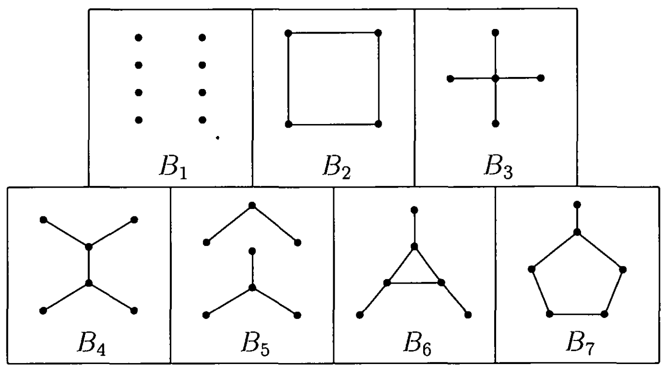


Figure 9.8: Minimal obstructions of the toroidal biembedding of the STS(7)s.

Proposition 9.1.6 *There are exactly seven maximal graphs and two maximum graphs that can be represented in the toroidal biembedding of the $STS(7)$s.*

Proof There are 65 graphs with seven edges. Proving that each graph is representable or not is a laborious task and quite unnecessary in this case. We only need to consider the 16 maximum graphs representable in the STS(7), which we already found and proved in Chapter 7, since if a graph is represented in the biembedding of the two STS(7)s then it is also represented in the STS(7). Using the listing in [51], these graphs are G116, G123, G124, G127, G130, G328, G330, G336, G338, G339, G340, G345, G349, G351, G352, G353. Apart from the graphs G338 and G353, the rest are non-representable in the K_7 triangulation. The graphs G116, G123, G124, G330 and G345 contain the 4-cycle as a subgraph, the graphs G130 and G339 contain the minimal obstruction B_4, the graphs G336, G349 and G352 contain B_5, the graph G328 contains B_6 and finally the graphs G127, G340 and G351 contain B_7. Therefore, there are just two maximum representable graphs and these are illustrated in Figure 9.9.

In order to find the maximal representable graphs we must first consider the graphs on six edges; there are 41 such graphs. We know from the proof of Proposition 9.1.5 that 30 of these graphs are non-representable so we may only consider the remaining 11: G97, G102, G105, G106, G277, G279, G283, G284, G286, G287 and G289. However, the graphs G97, G279, G283 and G286 are subgraphs of G338 which is maximum, and therefore are not maximal. The other 7 graphs are maximal. Similarly, there are 12 representable graphs with five edges: G35, G36, G38, G80 to G84, G244, G246, G247 and G248. All of these are subgraphs of the maximal and maximum graphs found above; G35 and G80 of G338, G36, G81, G82 and G83 of G102, G38 and G246 of G289. G84 of G106, G244 of G277 and G247 and G248 of G287. Finally, representable graphs with edges less than five are subgraphs of the maximal and maximum graphs found above. Hence, there are only seven maximal graphs. These are illustrated in Figure 9.10. ■

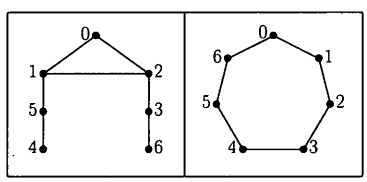


Figure 9.9: Maximum representable graphs in the triangulation of K_7.

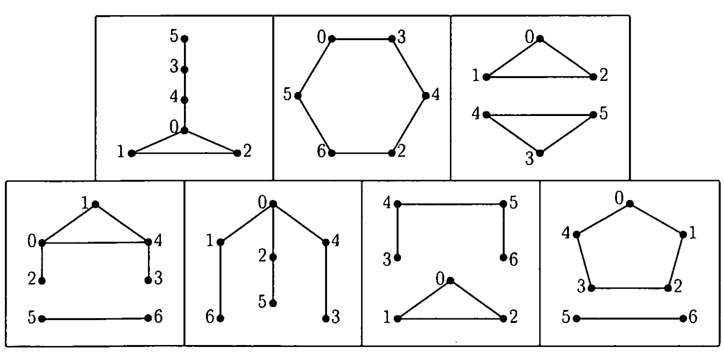


Figure 9.10: Maximal representable graphs in the triangulation of K_7.

We mentioned in the beginning of the chapter that the triangulations impose extra restrictions to representations. The previous proposition is a very clear example of this statement. There are 16 maximum graphs representable in the STS(7) but only two of these are representable in the triangulation of K_7.

We now turn our attention to regular graphs and specifically of regular graphs of degree 2.

9.2 Cycles in triangulations

In Chapter 7, we proved that every cycle of order n can be represented in a Steiner triple system of order $m \geq n$ except for $(n, m) = (3, 3)$. In this section we prove a similar result for triangulations of K_m. To be exact, we prove that every cycle of length n is representable in a triangular embedding of K_m, $m \geq n$, i.e. no two edges in the cycle are incident with the same triangular face, except for $(n, m) = (3, 3), (3, 4), (4, 4), (4, 6), (4, 7), (6, 6)$. We start with the easy case, $n = m - 1$.

Lemma 9.2.1 *Every cycle C_n, $n \geq 4$, can be represented in a triangulation of K_m, where $m = n + 1$.*

Proof Consider the rotation about any point of K_m. The points in this rotation form a cycle of length $m - 1$. Clearly, each edge of this cycle is incident to distinct triangles and therefore a valid representation is achieved. Note that the proof fails if $n = 3$. ∎

The next case is cycles of length up to $m - 6$. To prove this we use the same technique used in Theorem 7.3.1.

Lemma 9.2.2 *Every cycle C_n can be represented in a triangulation of K_m, where $m \geq n + 6$.*

Proof The proof is by induction on n. To start the induction at $n = 3$, pick any three points x, y and z which do not form a surface triangle. These three points represent C_3.

For the inductive step, suppose that we have a representation of C_n, say by points $1, 2, \ldots, n$ in cyclic order. Pick a point x that is not equal to any of

$\{1, 2, \ldots, n\} \cup t_i(1,2) \cup t_i(n-1, n) \cup t_i(1, n), i = 1, 2$. Such a choice is possible provided $m > n + 6$. Now remove the edge $\{1, n\}$ from C_n and add the edges $\{1, x\}, \{x, n\}$. Using the blocks with these edges gives a representation of C_{n+1}, in particular since $x \neq t_i(1, 2)$, the blocks containing $1, 2$ and the blocks containing $1, x$ are distinct. The same holds for the blocks containing the pairs x, n and $n, n-1$. Furthermore, the blocks containing $1, x$ and x, n are also distinct. Note that the inductive step breaks down when $m = n + 6$. ∎

The cases left to consider are $m = n$ and $m - 5 \leq n \leq m - 2$. The proofs for both of these cases use similar techniques but we require the extra assumption that $m \geq 12$.

Lemma 9.2.3 *Let $n \geq 12$. Then every cycle C_n can be represented in a triangulation of K_n.*

Proof Consider the cycle $(0, 1, 2, \ldots, m-2)$ of length $m-1$ forming the rotation at any vertex ∞. From this we may represent a cycle $C_{\infty,0}$ of length m: $(0, 1, 5, \infty, 2, 4, 3, 6, 7, \ldots, m-2)$ (Figure 9.11) which will fail to satisfy our requirement if and only if one or more of the following triples form a face in the embedding: $\{0, 1, 5\}$, $\{3, 4, 6\}$, $\{3, 6, 7\}$. We call these the critical triples of the cycle $C_{\infty,0}$.

Now consider the effect of rotating the cycle $C_{\infty,0}$. If we rotate one place we get the cycle $C_{\infty,1} = (1, 2, 6, \infty, 3, 5, 4, 7, 8, \ldots, 0)$. This also fails if one of the following critical triples forms a face: $\{1, 2, 6\}$, $\{4, 5, 7\}$, $\{4, 7, 8\}$. Note that these triples are distinct from the previous ones. By repeating the rotation we fail to achieve a representable cycle if in each position one of the three critical triples forms a face.

In addition to rotating the cycle we may reflect it to get $C'_{\infty,0} = (0, m-2, m-6, \infty, m-3, m-5, m-4, m-7, m-8, \ldots, 1)$. Again there are three critical triples, each of which may form a face. By rotating we get further cycles and we fail to achieve a representable cycle if in each position one of the three critical triples forms a face.

The cycles $C_{\infty,i}$ and $C'_{\infty,i}$, $i \in \mathbb{Z}_{m-1}$, will give distinct critical triples if and only if $m \geq 12$. The reason for this is simple. Without loss of generality consider the critical triples of $C_{\infty,0}$ and $C'_{\infty,0}$: $\{0,1,5\}$, $\{3,4,6\}$, $\{3,6,7\}$ and $\{0,m-2,m-6\}$, $\{m-4,m-5,m-7\}$, $\{m-4,m-7,m-8\}$ respectively. These lie in orbits under $\mathbb{Z}_{m-1}$ with base blocks $\{0,1,5\}$, $\{0,1,3\}$, $\{0,1,m-4\}$, $\{0,1,m-5\}$, $\{0,1,m-3\}$ and $\{0,1,4\}$ which are all distinct if and only if $m \geq 12$. Therefore, assuming $m \geq 12$, all the $2(m-1)$ cycles $C_{\infty,i}$ and $C'_{\infty,i}$ give distinct critical triples. For none of these cycles to be representable, at least $2(m-1)$ of the critical triples must form faces. If none of these cycles is representable, we say that the vertex ∞ is bad.

Now consider the effect of varying the vertex ∞. There are m choices for this vertex. If all vertices were bad then we should have a collection of at least $m \times 2(m-1)$ critical triples that form faces. A face can appear as a critical triple at most once for each of its three edges. So there would be at least $2m(m-1)/3$ distinct faces in the embedding. But the number of faces in the embedding is $m(m-1)/3$ which is a contradiction. Hence, not all vertices can be bad, and hence there is at least one representable cycle. ■

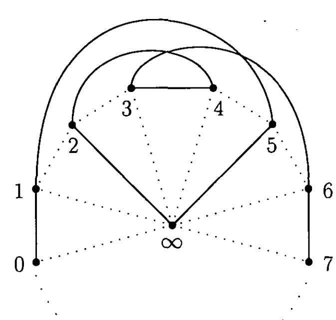


Figure 9.11: The cycle $C_{\infty,0}$

Lemma 9.2.4 *Assume $m \geq 12$. Then every cycle C_n can be represented in a triangulation of K_m, where $m-5 \leq n \leq m-2$.*

Proof Similarly to the proof of Lemma 9.2.3, we consider the cycles $C_{u,i}$ and $C'_{u,i}$, $u \in V(K_m)$, $i \in \mathbb{Z}_{m-1}$, for each case and reach a contradiction by counting the number of critical triples. However, in this proof the cycle $C'_{u,i}$ is not a reflection of the cycle $C_{u,i}$ but a different way of representing the cycle of that order. Below we give the cycles $C_{\infty,0}$ and $C'_{\infty,0}$ and their respective critical triples and base blocks for each case.

• $n = m - 5$

$C_{\infty,0} = (0, 4, \infty, 6, 8, 9, 10, \ldots, m-2)$ / $\{0, 4, m-2\}, \{6, 8, 9\}$ / $\{0, 1, 5\}, \{0, 1, m-3\}$

$C'_{\infty,0} = (0, 3, \infty, 5, 8, 9, 10, \ldots, m-2)$ / $\{0, 3, m-2\}, \{5, 8, 9\}$ / $\{0, 1, 4\}, \{0, 1, m-4\}$

• $n = m - 4$

$C_{\infty,0} = (0, 4, 5, 6, 7, 8, \ldots, m-2)$ / $\{0, 4, m-2\}, \{0, 4, 5\}$ / $\{0, 1, 5\}, \{0, 1, m-5\}$

$C'_{\infty,0} = (0, 1, 4, \infty, 6, 8, 9, 10 \ldots, m-2)$ / $\{0, 1, 4\}, \{6, 8, 9\}$ / $\{0, 1, 4\}, \{0, 1, m-3\}$

• $n = m - 3$

$C_{\infty,0} = (0, 3, 4, 5, \ldots, m-2)$ / $\{0, 3, m-2\}, \{0, 3, 4\}$ / $\{0, 1, 4\}, \{0, 1, m-4\}$

$C'_{\infty,0} = (0, 1, 5, \infty, 2, 7, \ldots, m-2)$ / $\{0, 1, 5\}, \{2, 7, 8\}$ / $\{0, 1, 5\}, \{0, 1, m-6\}$

• $n = m - 2$

$C_{\infty,0} = (0, 2, 3, 4, \ldots, m-2)$ / $\{0, 2, m-2\}, \{0, 2, 3\}$ / $\{0, 1, 3\}, \{0, 1, m-3\}$

$C'_{\infty,0} = (0, 1, 5, \infty, 2, 6, \ldots, m-2)$ / $\{0, 1, 5\}, \{2, 6, 7\}$ / $\{0, 1, 5\}, \{0, 1, m-5\}$

Assuming $m \geq 12$ for each case, all the $2(m-1)$ cycles $C_{\infty,i}$ and $C'_{\infty,i}$ give distinct critical triples. Using the same argument as in the proof of Lemma 9.2.3, we deduce that there is at least one representable cycle for each case. ■

Based on the above four lemmas we can state the following theorem.

Theorem 9.2.5 *Every cycle C_n can be represented in a triangulation of K_m, where $m \geq 12$ and $n \leq m$.*

To conclude this section we need to consider triangulations of order less than 12. Triangulations of K_m, $m < 12$, exist for $m = 3, 4, 6, 7, 9, 10$. From the first section of this chapter we know that no cycle can be represented in the triangulations of K_3 and K_4. We also know that the 4-cycle and the 6-cycle cannot be represented in the triangulation of K_6 and the 4-cycle cannot be represented in the triangulation of K_7. So this leaves only the triangulations of K_9 and K_{10}

to consider. There are precisely two triangulations of K_9 in a surface. These correspond to the twofold triple systems #35 and #36 of the listing in [8] of the 36 nonisomorphic TTS(9)s; #35 is not face two-colourable whereas #36 is face two-colourable. The two systems and the representable cycles are given below.

#35:
000000001111112222233344
112345672345673345645655
234567885468787876886778

$C_3 : 0\ 1\ 4$
$C_4 : 0\ 1\ 4\ 5$
$C_5 : 0\ 1\ 4\ 5\ 6$
$C_6 : 0\ 1\ 4\ 5\ 6\ 7$
$C_7 : 0\ 1\ 4\ 2\ 3\ 5\ 8$
$C_8 : 1\ 2\ 4\ 6\ 8\ 7\ 5\ 3$
$C_9 : 0\ 1\ 6\ 8\ 7\ 5\ 2\ 3\ 4$

#36:
000000001111112222233344
112345672345673345645655
234567885468787886776878

$C_3 : 0\ 1\ 4$
$C_4 : 0\ 1\ 4\ 5$
$C_5 : 0\ 1\ 4\ 5\ 6$
$C_6 : 0\ 1\ 4\ 5\ 6\ 7$
$C_7 : 0\ 1\ 4\ 2\ 3\ 5\ 8$
$C_8 : 1\ 2\ 4\ 6\ 8\ 7\ 5\ 3$
$C_9 : 0\ 1\ 6\ 8\ 7\ 5\ 2\ 3\ 4$

Finally, there are 394 nonisomorphic TTS(10)s without repeated blocks [6] 14 of which can be embedded [1, 5]. Using the listing in [1], Appendix E lists the cycle representations.

Therefore, considering all of the above results, we state the main theorem.

Theorem 9.2.6 *Every cycle C_n, $n \geq 3$, can be represented in every triangulation of K_m, $m \geq n$, except for $(n, m) = (3,3), (3,4), (4,4), (4,6), (4,7), (6,6)$.*

APPENDIX A

The 36 nonisomorphic TTS(9)s

1.

000000011111122222223344
113355773344663344556655
224466885577888866777788

2.

000000011111122222223344
113355773344663344556655
224466885578787866787878

3.

000000011111122222223344
113355673344673344556655
224467885568787867687878

4.

000000011111122222223344
113355773344663344555656
224466885758786768788877

5.

000000011111122222223344
113355773344663344555656
224466885758786867787887

6.

000000011111122222223344
113355673344673344555656
224467885657887868678787

7.

000000011111122222223344
113355673344673344555656
224467885658787867687887

8.

000000011111122222223344
113355673344673344555656
224467885658787867688778

9.

000000011111122222223344
113355673344663344555756
224467885858776767886878

10.

000000011111122222223344
113355773344563344565656
224466885768786857877887

11.

000000011111122222223344
113355673344573344565656
224467885678687856788787

12.

000000011111122222223344
113355673344573344565656
224467885678687856877887

13.
000000001111112222223344
113355673344573344565656
224467885867686758788787

14.
000000001111112222223344
113355673344573344565656
224467885867686758877887

15.
000000001111112222223344
113346673345573344555656
224557884676887868677887

16.
000000001111112222223344
113345673345673344556655
224567884586787867687878

17.
000000001111112222223344
113345673345673344555656
224567884856786778687887

18.
000000001111112222223344
113345673345663344555756
224567884857786778686887

19.
000000001111112222223344
113345673345663344555657
224567884857787867686788

20.
000000001111112222223344
113345673345563344566755
224567884876785678877868

21.
000000001111112222223344
113345673345563344576655
224567884876785867687788

22.
000000001111112222223344
113345673345563344575656
224567884786876857687887

23.
000000001111112222223344
113345673345563344565756
224567884876786758876887

24.
000000001111112222233333
112445672445674455644556
233567883567887868778687

25.
000000001111112222233333
112445672445674455644556
233567883568787867878687

26.
000000001111112222233334
112445672345673455644565
233567884567888768778678

27.
000000001111112222233334
112445672345673455644565
233567884568787868778687

28.
000000001111112222233334
112445672345673455644565
233567884568788767878678

29.
000000001111112222233334
112445672345673455644565
233567884576886878778876

30.
000000001111112222233334
112445672345673455644565
233567884576888678778678

31.
0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 2 2 3 3 3 3 4
1 1 2 4 4 5 6 7 2 3 4 5 6 7 3 4 5 5 6 4 4 5 6 5
2 3 3 5 6 7 8 8 4 5 8 6 7 8 7 7 6 8 8 6 8 8 7 7

32.
0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 2 2 3 3 3 3 4
1 1 2 4 4 5 6 7 2 3 4 5 6 7 3 4 5 5 6 4 4 5 6 5
2 3 3 5 6 7 8 8 4 5 8 6 7 8 8 7 6 8 7 6 7 7 8 8

33.
0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 2 2 3 3 3 4 4
1 1 2 3 5 5 6 7 2 3 4 4 6 7 3 3 4 5 6 4 5 6 5 5
2 3 4 4 6 7 8 8 5 5 6 8 7 8 7 8 6 8 7 7 6 8 7 8

34.
0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 2 2 3 3 3 4 4
1 1 2 3 5 5 6 7 2 3 4 4 6 7 3 3 4 5 6 4 5 5 5 6
2 3 4 4 6 7 8 8 5 6 5 8 7 8 6 7 7 8 8 8 7 8 6 7

35.
0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 2 2 3 3 3 4 4
1 1 2 3 4 5 6 7 2 3 4 5 6 7 3 3 4 5 6 4 5 6 5 5
2 3 4 5 6 7 8 8 5 4 6 8 7 8 7 8 7 6 8 8 6 7 7 8

36.
0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 2 2 3 3 3 4 4
1 1 2 3 4 5 6 7 2 3 4 5 6 7 3 3 4 5 6 4 5 6 5 5
2 3 4 5 6 7 8 8 5 4 6 8 7 8 7 8 8 6 7 7 6 8 7 8

APPENDIX B

Rotation schemes of the TTS(9) embeddings

1.
0 : 1 2 | 3 4 | 5 6 | 7 8 |
1 : 2 0 | 3 5 | 4 7 | 6 8 |
2 : 0 1 | 3 8 | 4 6 | 5 7 |
3 : 4 0 | 1 5 | 2 8 | 6 7 |
4 : 0 3 | 1 7 | 2 6 | 5 8 |
5 : 6 0 | 1 3 | 2 7 | 4 8 |
6 : 0 5 | 1 8 | 2 4 | 3 7 |
7 : 8 0 | 1 4 | 2 5 | 3 6 |
8 : 0 7 | 1 6 | 2 3 | 4 5 |

2.
0 : 1 2 | 3 4 | 5 6 | 7 8 |
1 : 2 0 | 3 5 | 4 7 6 8 |
2 : 0 1 | 3 7 5 8 | 4 6 |
3 : 4 0 | 1 5 | 8 6 7 2 |
4 : 0 3 | 8 5 7 1 | 2 6 |
5 : 6 0 | 1 3 | 2 7 4 8 |
6 : 0 5 | 1 7 3 8 | 2 4 |
7 : 8 0 | 1 4 5 2 3 6 |
8 : 0 7 | 6 3 2 5 4 1 |

3.
0 : 1 2 | 3 4 | 5 6 8 7 |
1 : 2 0 | 3 5 | 8 7 6 4 |
2 : 0 1 | 3 7 4 6 5 8 |
3 : 4 0 | 1 5 | 8 6 7 2 |
4 : 0 3 | 1 6 2 7 5 8 |
5 : 6 0 7 4 8 2 | 1 3 |
6 : 0 5 2 4 1 7 3 8
7 : 0 5 4 2 3 6 1 8
8 : 0 6 3 2 5 4 1 7

4.
0 : 1 2 | 3 4 | 5 6 | 7 8 |
1 : 2 0 | 3 5 4 8 6 7 |
2 : 0 1 | 7 5 8 4 6 3 |
3 : 4 0 | 7 2 6 8 5 1 |
4 : 0 3 | 1 5 7 6 2 8 |
5 : 6 0 | 1 3 8 2 7 4 |
6 : 0 5 | 8 3 2 4 7 1 |
7 : 8 0 | 6 4 5 2 3 1 |
8 : 0 7 | 1 4 2 5 3 6 |

5.
0 : 1 2 | 3 4 | 5 6 | 7 8 |
1 : 2 0 | 3 5 4 8 6 7 |
2 : 0 1 | 8 5 7 4 6 3 |
3 : 4 0 | 7 5 1 | 2 6 8 |
4 : 0 3 | 1 5 8 | 7 6 2 |
5 : 6 0 | 1 3 7 2 8 4 |
6 : 0 5 | 8 3 2 4 7 1 |
7 : 8 0 | 6 4 2 5 3 1 |
8 : 0 7 | 1 4 5 2 3 6 |

6.
0 : 1 2 | 3 4 | 5 6 8 7 |
1 : 2 0 | 3 5 4 7 8 6 |
2 : 0 1 | 8 4 6 5 7 3 |
3 : 4 0 | 6 7 2 8 5 1 |
4 : 0 3 | 7 6 2 8 5 1 |
5 : 6 0 7 2 | 1 3 8 4 |
6 : 0 5 2 4 7 3 1 8
7 : 0 5 2 3 6 4 1 8
8 : 0 6 1 7 | 2 3 5 4 |

7.

0 : 1 2 | 3 4 | 5 6 8 7 |
1 : 2 0 | 6 7 8 4 5 3 |
2 : 0 1 | 3 7 4 6 5 8 |
3 : 4 0 | 1 5 7 2 8 6 |
4 : 0 3 | 8 5 1 | 7 6 2 |
5 : 6 0 7 3 1 4 8 2
6 : 0 5 2 4 7 1 3 8
7 : 8 1 6 4 2 3 5 0
8 : 0 6 3 2 5 4 1 7

8.

0 : 1 2 | 3 4 | 5 6 8 7 |
1 : 2 0 | 3 5 4 8 7 6 |
2 : 0 1 | 3 7 4 6 5 8 |
3 : 4 0 | 6 7 2 8 5 1 |
4 : 0 3 | 1 5 7 2 6 8 |
5 : 6 0 7 4 1 3 8 2
6 : 0 5 2 4 8 | 1 3 7 |
7 : 8 1 6 3 2 4 5 0
8 : 0 6 4 1 7 | 5 3 2 |

9.

0 : 1 2 | 3 4 | 5 6 8 7 |
1 : 2 0 | 3 5 4 8 | 6 7 |
2 : 0 1 | 3 6 4 7 | 5 8 |
3 : 4 0 | 8 7 2 6 5 1 |
4 : 0 3 | 1 5 7 2 6 8 |
5 : 6 0 7 4 1 3 | 2 8 |
6 : 0 5 3 2 4 8 | 1 7 |
7 : 8 3 2 4 5 0 | 1 6 |
8 : 0 6 4 1 3 7 | 2 5 |

10.

0 : 1 2 | 3 4 | 5 6 | 7 8 |
1 : 2 0 | 3 5 7 | 4 6 8 |
2 : 0 1 | 3 6 7 4 5 8 |
3 : 4 0 | 7 5 1 | 8 6 2 |
4 : 0 3 | 8 5 2 7 6 1 |
5 : 6 0 | 1 3 7 | 2 4 8 |
6 : 0 5 | 1 4 7 2 3 8 |
7 : 8 0 | 5 3 1 | 6 4 2 |
8 : 0 7 | 6 3 2 5 4 1 |

11.

0 : 1 2 | 3 4 | 5 6 8 7 |
1 : 2 0 | 6 5 3 | 4 7 8 |
2 : 0 1 | 8 6 4 5 7 3 |
3 : 4 0 | 1 5 8 2 7 6 |
4 : 0 3 | 8 5 2 6 7 1 |
5 : 6 0 7 2 4 8 3 1
6 : 0 5 1 3 7 4 2 8
7 : 8 1 4 6 3 2 5 0
8 : 0 6 2 3 5 4 1 7

12.

0 : 1 2 | 3 4 | 7 8 6 5 |
1 : 2 0 | 3 5 6 | 8 7 4 |
2 : 0 1 | 8 5 4 6 7 3 |
3 : 4 0 | 6 8 2 7 5 1 |
4 : 0 3 | 1 7 6 2 5 8 |
5 : 1 3 7 0 6 | 8 4 2 |
6 : 8 3 1 5 0 | 2 4 7 |
7 : 0 5 3 2 6 4 1 8
8 : 7 1 4 5 2 3 6 0

13.

0 : 1 2 | 3 4 | 5 6 8 7 |
1 : 2 0 | 8 7 4 6 5 3 |
2 : 0 1 | 3 6 8 4 5 7 |
3 : 4 0 | 1 5 8 | 2 6 7 |
4 : 0 3 | 7 6 1 | 8 5 2 |
5 : 6 0 7 2 4 8 3 1
6 : 0 5 1 4 7 3 2 8
7 : 8 1 4 6 3 2 5 0
8 : 7 1 3 5 4 2 6 0

14.

0 : 1 2 | 3 4 | 5 6 8 7 |
1 : 2 0 | 8 7 4 6 5 3 |
2 : 0 1 | 3 6 7 | 4 5 8 |
3 : 4 0 | 8 6 2 7 5 1 |
4 : 0 3 | 7 6 1 | 2 5 8 |
5 : 6 0 7 3 1 | 2 4 8 |
6 : 0 5 1 4 7 2 3 8
7 : 0 5 3 2 6 4 1 8
8 : 0 6 3 1 7 | 2 4 5 |

15.
0 : 1 2 | 3 4 5 | 6 7 8 |
1 : 2 0 | 6 5 8 7 4 3 |
2 : 0 1 | 3 7 5 6 4 8 |
3 : 4 0 5 7 2 8 6 1
4 : 0 3 1 7 6 2 8 5
5 : 4 8 1 6 2 7 3 0
6 : 4 2 5 1 3 8 0 7
7 : 0 6 4 1 8 | 2 3 5 |
8 : 0 6 3 2 4 5 1 7

16.
0 : 1 2 | 3 4 6 8 7 5 |
1 : 2 0 | 5 6 7 8 4 3 |
2 : 0 1 | 8 5 6 4 7 3 |
3 : 4 0 5 1 | 2 7 6 8 |
4 : 0 3 1 8 5 7 2 6
5 : 7 4 8 2 6 1 3 0
6 : 0 4 2 5 1 7 3 8
7 : 8 1 6 3 2 4 5 0
8 : 0 6 3 2 5 4 1 7

17.
0 : 1 2 | 3 4 6 8 7 5 |
1 : 2 0 | 8 7 6 5 4 3 |
2 : 0 1 | 7 4 8 5 6 3 |
3 : 4 0 5 7 2 6 8 1
4 : 0 3 1 5 8 2 7 6
5 : 7 3 0 | 6 2 8 4 1 |
6 : 8 3 2 5 1 7 4 0
7 : 8 1 6 4 2 3 5 0
8 : 7 1 3 6 0 | 2 4 5 |

18.
0 : 1 2 | 3 4 6 8 7 5 |
1 : 2 0 | 8 6 7 5 4 3 |
2 : 0 1 | 3 6 5 8 4 7 |
3 : 4 0 5 6 2 7 8 1
4 : 0 3 1 5 8 2 7 6
5 : 7 1 4 8 2 6 3 0
6 : 0 4 7 1 8 | 2 3 5 |
7 : 8 3 2 4 6 1 5 0
8 : 0 6 1 3 7 | 2 4 5 |

19.
0 : 1 2 | 3 4 6 8 7 5 |
1 : 2 0 | 8 6 7 5 4 3 |
2 : 0 1 | 3 7 4 6 5 8 |
3 : 4 0 5 6 7 2 8 1
4 : 0 3 1 5 8 7 2 6
5 : 7 1 4 8 2 6 3 0
6 : 8 1 7 3 5 2 4 0
7 : 8 4 2 3 6 1 5 0
8 : 0 6 1 3 2 5 4 7

20.
0 : 1 2 | 3 4 6 8 7 5 |
1 : 2 0 | 8 6 5 7 4 3 |
2 : 0 1 | 3 5 8 4 7 6 |
3 : 4 0 5 2 6 7 8 1
4 : 0 3 1 7 2 8 5 6
5 : 7 1 6 4 8 2 3 0
6 : 0 4 5 1 8 | 7 3 2 |
7 : 8 3 6 2 4 1 5 0
8 : 0 6 1 3 7 | 5 4 2 |

21.
0 : 1 2 | 3 4 6 8 7 5 |
1 : 2 0 | 8 6 5 7 4 3 |
2 : 0 1 | 3 5 6 4 7 8 |
3 : 4 0 5 2 8 1 | 6 7 |
4 : 0 3 1 7 2 6 | 5 8 |
5 : 7 1 6 2 3 0 | 4 8 |
6 : 0 4 2 5 1 8 | 3 7 |
7 : 8 2 4 1 5 0 | 3 6 |
8 : 0 6 1 3 2 7 | 4 5 |

22.
0 : 1 2 | 3 4 6 8 7 5 |
1 : 2 0 | 7 6 5 8 4 3 |
2 : 0 1 | 8 7 4 5 6 3 |
3 : 4 0 5 7 1 | 2 6 8 |
4 : 0 3 1 8 5 2 7 6
5 : 0 3 7 | 1 6 2 4 8 |
6 : 0 4 7 1 5 2 3 8
7 : 0 5 3 1 6 4 2 8
8 : 0 6 3 2 7 | 5 4 1 |

23.
0 : 1 2 | 3 4 6 8 7 5 |
1 : 2 0 | 8 6 5 7 4 3 |
2 : 0 1 | 3 6 7 | 4 5 8 |
3 : 4 0 5 6 2 7 8 1
4 : 0 3 1 7 6 | 8 5 2 |
5 : 7 1 6 3 0 | 2 4 8 |
6 : 0 4 7 2 3 5 1 8
7 : 8 3 2 6 4 1 5 0
8 : 0 6 1 3 7 | 5 4 2 |

24.
0 : 1 2 3 | 4 5 7 8 6 |
1 : 2 0 3 | 6 8 7 5 4 |
2 : 0 1 3 | 4 7 6 5 8 |
3 : 2 1 0 | 8 5 6 7 4 |
4 : 5 0 6 1 | 8 3 7 2 |
5 : 0 4 1 7 | 2 6 3 8 |
6 : 8 1 4 0 | 7 3 5 2 |
7 : 0 5 1 8 | 2 4 3 6 |
8 : 7 1 6 0 | 5 3 4 2 |

25.
0 : 1 2 3 | 4 5 7 8 6 |
1 : 2 0 3 | 6 7 8 5 4 |
2 : 0 1 3 | 4 7 5 6 8 |
3 : 0 1 2 | 4 7 6 5 8 |
4 : 5 0 6 1 | 2 7 3 8 |
5 : 0 4 1 8 3 6 2 7
6 : 8 2 5 3 7 1 4 0
7 : 0 5 2 4 3 6 1 8
8 : 0 6 2 4 3 5 1 7

26.
0 : 1 2 3 | 4 5 7 8 6 |
1 : 2 0 3 5 7 8 6 4
2 : 0 1 4 7 6 5 8 3
3 : 2 8 4 7 6 5 1 0
4 : 8 3 7 2 1 6 0 5
5 : 7 1 3 6 2 8 4 0
6 : 0 4 1 8 | 7 3 5 2 |
7 : 8 1 5 0 | 2 4 3 6 |
8 : 0 6 1 7 | 2 3 4 5 |

27.
0 : 1 2 3 | 4 5 7 8 6 |
1 : 2 0 3 5 8 7 6 4
2 : 0 1 4 8 5 6 7 3
3 : 2 7 4 8 6 5 1 0
4 : 7 3 8 2 1 6 0 5
5 : 0 4 7 | 1 3 6 2 8 |
6 : 0 4 1 7 2 5 3 8
7 : 0 5 4 3 2 6 1 8
8 : 7 1 5 2 4 3 6 0

28.
0 : 1 2 3 | 4 5 7 8 6 |
1 : 2 0 3 5 8 7 6 4
2 : 0 1 4 7 5 6 8 3
3 : 2 8 4 7 6 5 1 0
4 : 8 3 7 2 1 6 0 5
5 : 0 4 8 1 3 6 2 7
6 : 0 4 1 7 3 5 2 8
7 : 0 5 2 4 3 6 1 8
8 : 7 1 5 4 3 2 6 0

29.
0 : 1 2 3 | 4 5 7 8 6 |
1 : 2 0 3 5 6 8 7 4
2 : 0 1 4 8 5 7 6 3
3 : 2 6 7 4 8 5 1 0
4 : 5 0 6 | 7 3 8 2 1 |
5 : 0 4 6 1 3 8 2 7
6 : 8 1 5 4 0 | 2 3 7 |
7 : 0 5 2 6 3 4 1 8
8 : 7 1 6 0 | 2 4 3 5 |

30.
0 : 1 2 3 | 4 5 7 8 6 |
1 : 2 0 3 5 6 8 7 4
2 : 0 1 4 6 7 5 8 3
3 : 2 8 4 7 6 5 1 0
4 : 5 0 6 2 1 7 3 8
5 : 0 4 8 2 7 | 1 3 6 |
6 : 8 1 5 3 7 2 4 0
7 : 0 5 2 6 3 4 1 8
8 : 7 1 6 0 | 5 4 3 2 |

31.
0 : 1 2 3 | 6 8 7 5 4 |
1 : 2 0 3 5 6 7 8 4
2 : 0 1 4 7 3 | 8 6 5 |
3 : 2 7 6 4 8 5 1 0
4 : 7 2 1 8 3 6 0 5
5 : 7 4 0 | 1 3 8 2 6 |
6 : 0 4 3 7 1 5 2 8
7 : 8 1 6 3 2 4 5 0
8 : 0 6 2 5 3 4 1 7

32.
0 : 1 2 3 | 4 5 7 8 6 |
1 : 2 0 3 5 6 7 8 4
2 : 0 1 4 7 6 5 8 3
3 : 2 8 6 4 7 5 1 0
4 : 5 0 6 3 7 2 1 8
5 : 0 4 8 2 6 1 3 7
6 : 0 4 3 8 | 1 5 2 7 |
7 : 8 1 6 2 4 3 5 0
8 : 0 6 3 2 5 4 1 7

33.
0 : 1 2 4 3 | 5 6 8 7 |
1 : 2 0 3 5 | 4 6 7 8 |
2 : 0 1 5 8 3 7 6 4
3 : 4 7 2 8 6 5 1 0
4 : 0 2 6 1 8 5 7 3
5 : 6 0 7 4 8 2 1 3
6 : 0 5 3 8 | 1 4 2 7 |
7 : 8 1 6 2 3 4 5 0
8 : 0 6 3 2 5 4 1 7

34.
0 : 1 2 4 3 | 5 6 8 7 |
1 : 2 0 3 6 7 8 4 5
2 : 0 1 5 8 6 3 7 4
3 : 4 8 5 7 2 6 1 0
4 : 0 2 7 6 5 1 8 3
5 : 6 0 7 3 8 2 1 4
6 : 0 5 4 7 1 3 2 8
7 : 8 1 6 4 2 3 5 0
8 : 0 6 2 5 3 4 1 7

35.
0 : 1 2 4 6 8 7 5 3
1 : 2 0 3 4 6 7 8 5
2 : 0 1 5 6 8 3 7 4
3 : 5 6 7 2 8 4 1 0
4 : 0 2 7 5 8 3 1 6
5 : 7 4 8 1 2 6 3 0
6 : 0 4 1 7 3 5 2 8
7 : 8 1 6 3 2 4 5 0
8 : 0 6 2 3 4 5 1 7

36.
0 : 1 2 4 6 8 7 5 3
1 : 2 0 3 4 6 7 8 5
2 : 0 1 5 6 7 3 8 4
3 : 5 6 8 2 7 4 1 0
4 : 0 2 8 5 7 3 1 6
5 : 7 4 8 1 2 6 3 0
6 : 0 4 1 7 2 5 3 8
7 : 8 1 6 2 3 4 5 0
8 : 0 6 3 2 4 5 1 7

APPENDIX C

Maximal complete bipartite graphs in the STS(15)s

#	$K_{1,7}$	$K_{2,6}$	$K_{3,5}$	$K_{3,6}$	$K_{4,4}$
1	1920	840	840	0	1050
2	1920	648	528	0	570
3	1920	552	372	0	330
4	1920	504	356	0	306
5	1920	504	368	0	370
6	1920	432	324	0	306
7	1920	408	360	0	450
8	1920	432	276	0	206
9	1920	396	268	0	170
10	1920	396	274	0	202
11	1920	356	264	4	178
12	1920	410	272	2	166
13	1920	408	268	0	194
14	1920	432	270	0	174
15	1920	360	258	0	202
16	1920	504	294	0	210
17	1920	360	264	0	234
18	1920	360	252	0	170
19	1920	320	272	8	238
20	1920	338	248	2	130
21	1920	344	254	2	130
22	1920	326	260	8	142
23	1920	324	231	0	104
24	1920	334	232	2	100
25	1920	338	230	2	112
26	1920	356	231	2	114
27	1920	304	229	2	108
28	1920	314	230	4	104
29	1920	332	230	2	100
30	1920	306	231	4	108
31	1920	320	234	0	136
32	1920	306	236	4	96
33	1920	298	228	4	86
34	1920	302	231	4	86
35	1920	308	227	2	82
36	1920	278	218	0	80
37	1920	270	246	0	96
38	1920	290	237	4	100
39	1920	302	232	2	86
40	1920	302	224	2	82
41	1920	298	228	2	86
42	1920	298	255	8	104
43	1920	282	225	0	96
44	1920	274	227	0	72
45	1920	288	234	2	84
46	1920	280	240	4	76
47	1920	288	233	2	78
48	1920	278	230	4	72
49	1920	276	237	4	76
50	1920	270	245	4	112
51	1920	294	239	6	84
52	1920	288	230	4	68
53	1920	288	231	4	78
54	1920	302	239	8	90
55	1920	294	239	6	84
56	1920	282	233	4	72
57	1920	262	241	4	84
58	1920	272	234	4	86
59	1920	320	239	8	82
60	1920	288	253	10	108
61	1920	308	266	14	154
62	1920	266	230	2	58
63	1920	266	239	2	106
64	1920	290	236	8	94
65	1920	280	240	4	76
66	1920	274	242	4	80
67	1920	272	249	4	84
68	1920	268	234	2	64
69	1920	268	244	6	84
70	1920	288	234	4	84
71	1920	262	237	2	68
72	1920	272	249	6	84
73	1920	288	266	8	128
74	1920	272	230	0	88
75	1920	282	249	8	108
76	1920	280	220	0	80
77	1920	252	246	2	48
78	1920	272	250	8	128
79	1920	264	258	0	192
80	1920	240	270	0	120

APPENDIX D

Number of occurrences of $K_{2,4}$ in the STS(15)s

#	$K_{2,4}$
1	11340
2	11244
3	11196
4	11172
5	11172
6	11136
7	11124
8	11136
9	11118
10	11118
11	11094
12	11121
13	11124
14	11136
15	11100
16	11172
17	11100
18	11100
19	11076
20	11085
21	11085
22	11076
23	11079
24	11082
25	11085
26	11094
27	11067
28	11070
29	11082
30	11067
31	11079
32	11064
33	11061
34	11061
35	11064
36	11055
37	11043
38	11052
39	11061
40	11064
41	11061
42	11049
43	11055
44	11049
45	11052
46	11046
47	11055
48	11049
49	11046
50	11043
51	11052
52	11052
53	11055
54	11058
55	11052
56	11049
57	11040
58	11049
59	11064
60	11046
61	11067
62	11046
63	11046
64	11055
65	11046
66	11043
67	11040
68	11043
69	11040
70	11052
71	11040
72	11040
73	11043
74	11049
75	11046
76	11055
77	11031
78	11043
79	11043
80	11025

APPENDIX E

Cycle representations in the 14 TTS(10) embeddings.

1.

0000000001111111222223334444555
1122334682233578335674675567 66
4578576994668799459899988978 78

$C_3: 0\ 1\ 2$
$C_4: 0\ 1\ 2\ 3$
$C_5: 0\ 1\ 2\ 3\ 6$
$C_6: 0\ 1\ 2\ 3\ 6\ 7$
$C_7: 0\ 1\ 2\ 3\ 6\ 7\ 8$
$C_8: 0\ 1\ 2\ 5\ 4\ 3\ 6\ 7$
$C_9: 0\ 1\ 2\ 5\ 4\ 3\ 6\ 7\ 8$
$C_{10}: 0\ 1\ 2\ 5\ 4\ 3\ 6\ 8\ 9\ 7$

2.

000000000111111122222333444567
112233456223345733556446556788
674758899485698989697797678899

$C_3: 0\ 1\ 2$
$C_4: 0\ 1\ 2\ 3$
$C_5: 0\ 1\ 2\ 3\ 4$
$C_6: 0\ 1\ 2\ 3\ 4\ 5$
$C_7: 0\ 1\ 2\ 3\ 4\ 5\ 8$
$C_8: 0\ 1\ 2\ 3\ 4\ 5\ 8\ 9$
$C_9: 0\ 1\ 2\ 3\ 4\ 6\ 9\ 5\ 8$
$C_{10}: 0\ 1\ 2\ 3\ 4\ 6\ 7\ 5\ 9\ 8$

3.

000000000111111122222333444557
112233568223346633456445567678
454767989578987968989579689789

$C_3: 0\ 1\ 2$
$C_4: 0\ 1\ 2\ 3$
$C_5: 0\ 1\ 2\ 3\ 9$
$C_6: 0\ 1\ 2\ 3\ 9\ 6$
$C_7: 0\ 1\ 2\ 3\ 9\ 6\ 7$
$C_8: 0\ 1\ 2\ 3\ 9\ 6\ 7\ 8$
$C_9: 0\ 1\ 2\ 3\ 9\ 6\ 7\ 4\ 8$
$C_{10}: 0\ 1\ 2\ 3\ 9\ 6\ 7\ 4\ 5\ 8$

4.

000000000111111122222333444555
112233456223344733458667667678
687847599675958945969898789789

$C_3 : 0\ 1\ 2$
$C_4 : 0\ 1\ 2\ 3$
$C_5 : 0\ 1\ 2\ 3\ 9$
$C_6 : 0\ 1\ 2\ 3\ 9\ 4$
$C_7 : 0\ 1\ 2\ 3\ 9\ 8\ 4$
$C_8 : 0\ 1\ 2\ 3\ 9\ 8\ 7\ 4$
$C_9 : 0\ 1\ 2\ 3\ 9\ 5\ 6\ 8\ 7$
$C_{10} : 0\ 1\ 2\ 3\ 9\ 5\ 6\ 8\ 7\ 4$

5.

000000000111111122222333444555
112233566223346633457448678677
454878979785798969569569789889

$C_3 : 0\ 1\ 2$
$C_4 : 0\ 1\ 2\ 3$
$C_5 : 0\ 1\ 2\ 4\ 3$
$C_6 : 0\ 1\ 2\ 4\ 3\ 9$
$C_7 : 0\ 1\ 2\ 4\ 3\ 9\ 7$
$C_8 : 0\ 1\ 2\ 4\ 3\ 9\ 7\ 8$
$C_9 : 0\ 1\ 2\ 4\ 3\ 9\ 7\ 8\ 6$
$C_{10} : 0\ 1\ 2\ 3\ 4\ 7\ 5\ 6\ 9\ 8$

6.

000000000111111122222233344456
112334457233456833456756755768
279586968548697946887979979889

$C_3 : 0\ 1\ 3$
$C_4 : 0\ 1\ 3\ 6$
$C_5 : 0\ 1\ 3\ 2\ 5$
$C_6 : 0\ 1\ 3\ 2\ 5\ 4$
$C_7 : 0\ 1\ 3\ 2\ 5\ 4\ 8$
$C_8 : 0\ 1\ 3\ 2\ 5\ 4\ 8\ 6$
$C_9 : 0\ 1\ 3\ 9\ 2\ 5\ 4\ 8\ 6$
$C_{10} : 0\ 1\ 3\ 2\ 5\ 4\ 8\ 6\ 7\ 9$

7.

000000000111111122222333444567
112233457223345633455455667688
894667598674758989978869789799

$C_3 : 0\ 1\ 2$
$C_4 : 0\ 1\ 2\ 3$
$C_5 : 0\ 1\ 2\ 3\ 4$
$C_6 : 0\ 1\ 2\ 3\ 4\ 6$
$C_7 : 0\ 1\ 2\ 3\ 4\ 5\ 6$
$C_8 : 0\ 1\ 2\ 3\ 4\ 5\ 8\ 6$
$C_9 : 0\ 1\ 2\ 3\ 4\ 9\ 5\ 8\ 6$
$C_{10} : 0\ 1\ 2\ 3\ 4\ 9\ 5\ 8\ 6\ 7$

8.

000000000111111122222233334467
112345568233456734455644565578
243697879859768976879978896989

$C_3 : 0\ 1\ 3$
$C_4 : 0\ 1\ 3\ 7$
$C_5 : 0\ 1\ 3\ 7\ 8$
$C_6 : 0\ 1\ 3\ 7\ 6\ 5$
$C_7 : 0\ 1\ 3\ 7\ 6\ 5\ 9$
$C_8 : 0\ 1\ 8\ 3\ 7\ 6\ 5\ 9$
$C_9 : 0\ 1\ 5\ 2\ 3\ 4\ 6\ 7\ 9$
$C_{10} : 0\ 1\ 8\ 5\ 2\ 3\ 4\ 6\ 7\ 9$

9.

000000000111111122222333444567
112233558223346633457445556678
494768679595877869688797898999

$C_3 : 0\ 1\ 2$
$C_4 : 0\ 1\ 2\ 3$
$C_5 : 0\ 1\ 2\ 3\ 5$
$C_6 : 0\ 1\ 2\ 3\ 4\ 5$
$C_7 : 0\ 8\ 1\ 2\ 3\ 4\ 5$
$C_8 : 0\ 8\ 1\ 2\ 3\ 4\ 6\ 7$
$C_9 : 0\ 8\ 1\ 2\ 3\ 4\ 5\ 6\ 7$
$C_{10} : 0\ 1\ 3\ 2\ 4\ 8\ 9\ 5\ 7\ 6$

10.

000000000111111122222333444566
112233456223345733457456558877
674589879894596867689798679989

$C_3 : 0\ 1\ 2$
$C_4 : 0\ 1\ 2\ 3$
$C_5 : 0\ 1\ 2\ 3\ 4$
$C_6 : 0\ 1\ 2\ 3\ 4\ 5$
$C_7 : 0\ 1\ 2\ 3\ 4\ 5\ 8$
$C_8 : 0\ 1\ 2\ 3\ 9\ 4\ 5\ 8$
$C_9 : 0\ 1\ 3\ 2\ 4\ 8\ 5\ 7\ 9$
$C_{10} : 0\ 1\ 3\ 2\ 4\ 8\ 6\ 5\ 7\ 9$

11.

000000000111111122222333444556
112233447223345633566455678787
795869568497856847789968789999

$C_3 : 0\ 1\ 2$
$C_4 : 0\ 1\ 2\ 3$
$C_5 : 0\ 1\ 2\ 3\ 5$
$C_6 : 0\ 1\ 2\ 3\ 6\ 8$
$C_7 : 0\ 1\ 2\ 3\ 6\ 4\ 8$
$C_8 : 0\ 1\ 2\ 3\ 6\ 4\ 5\ 8$
$C_9 : 0\ 1\ 2\ 3\ 9\ 6\ 4\ 5\ 8$
$C_{10} : 0\ 1\ 3\ 2\ 9\ 4\ 5\ 8\ 7\ 6$

12.

000000000111111122222333455566
112233445223344633447448567778
588967796695978757568689988999

$C_3 : 0\ 1\ 2$
$C_4 : 0\ 1\ 2\ 3$
$C_5 : 0\ 1\ 2\ 3\ 4$
$C_6 : 0\ 1\ 3\ 2\ 6\ 7$
$C_7 : 0\ 1\ 4\ 3\ 2\ 6\ 7$
$C_8 : 0\ 1\ 4\ 3\ 2\ 6\ 5\ 7$
$C_9 : 0\ 9\ 1\ 4\ 3\ 2\ 6\ 5\ 7$
$C_{10} : 0\ 8\ 9\ 1\ 4\ 3\ 2\ 6\ 5\ 7$

13.

000000000111111122222333455567
112233445223344633445446766878
686979578795758958687698979989

$C_3 : 0\ 1\ 2$
$C_4 : 0\ 1\ 2\ 3$
$C_5 : 0\ 1\ 2\ 3\ 4$
$C_6 : 0\ 1\ 3\ 2\ 4\ 9$
$C_7 : 0\ 1\ 3\ 2\ 4\ 5\ 9$
$C_8 : 0\ 1\ 3\ 4\ 5\ 6\ 8\ 2$
$C_9 : 0\ 1\ 3\ 4\ 5\ 6\ 8\ 2\ 7$
$C_{10} : 0\ 1\ 3\ 4\ 5\ 6\ 8\ 2\ 7\ 9$

14.

000000000111111122222333344455
112234577223456833466456756867
365849689477598959879688977998

$C_3 : 0\ 1\ 2$
$C_4 : 0\ 1\ 2\ 9$
$C_5 : 0\ 1\ 2\ 9\ 5$
$C_6 : 0\ 1\ 2\ 9\ 5\ 4$
$C_7 : 0\ 1\ 2\ 9\ 5\ 4\ 8$
$C_8 : 0\ 1\ 2\ 9\ 5\ 3\ 4\ 8$
$C_9 : 0\ 1\ 9\ 2\ 4\ 3\ 5\ 6\ 7$
$C_{10} : 0\ 1\ 9\ 4\ 3\ 5\ 6\ 8\ 2\ 7$

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